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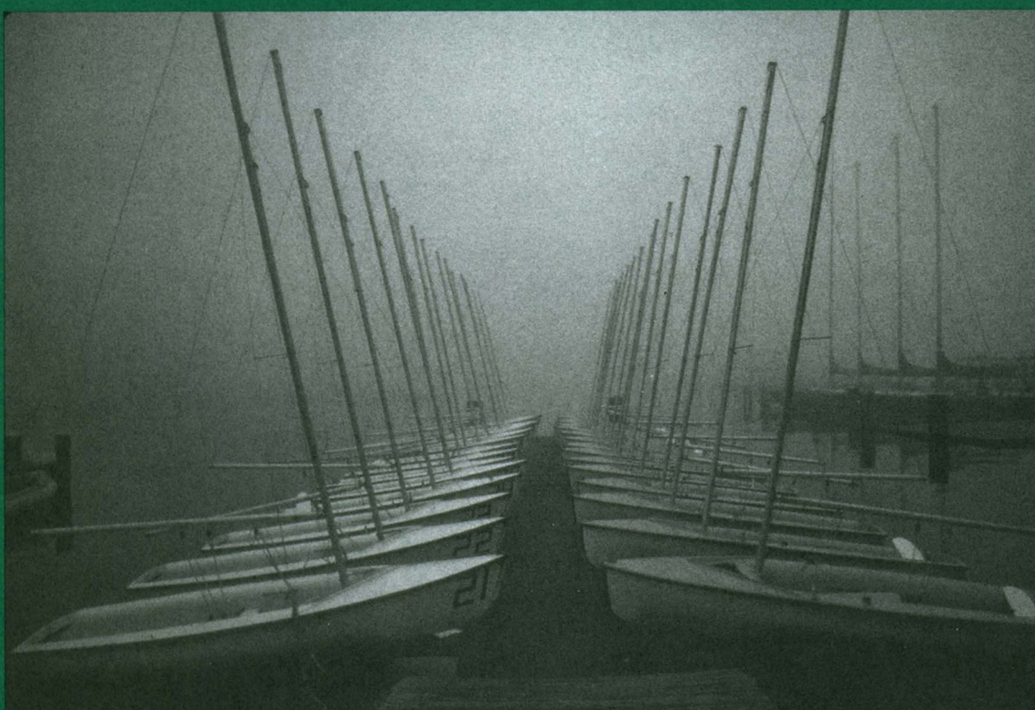
MATHEMATICS MAGAZINE



The River Crossing Game

- A Fresh Look at Peg Solitaire
- Counting Cyclic Binary Strings
- A Sequence of Polynomials Related to the Evaluation of the Riemann Zeta Function
- Brussells Sprouts and Cloves

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Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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The **cover image:** see page 15.

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The River Crossing Game

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Games of chance are often introduced in elementary and middle school classrooms as a way of motivating lessons on probability and statistics. One interesting game that appears in a curriculum used by local middle schools is called the River Crossing Game [**1, 2**]. It is a game for two players involving the sum of two dice which can be learned by school children in a few minutes. Yet an effort to understand the game mathematically yields a number of interesting and counterintuitive results, to the point that one may wonder exactly what children playing this game might be expected to learn!

Rules of the game

On each side of a river are docks numbered from 1 to 12. The players are each given twelve chips (boats) to place as they see fit at the docks. They decide upon the initial placement of their chips without knowledge of the placement of their opponent's chips. This can be accomplished either by placing a barrier in the middle of the game board or by having each player record his starting position on a piece of paper prior to actually placing the chips on the board. FIGURE 1 shows an example of two players' initial placement of chips along opposite sides of the river.

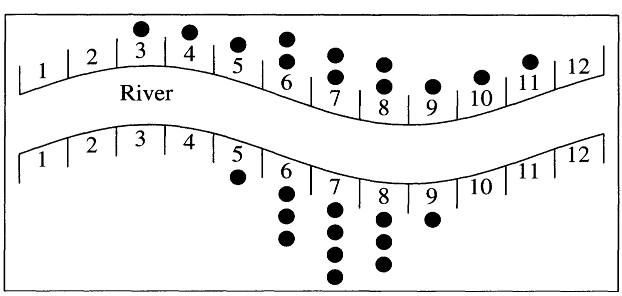

Figure 1 Game board with starting positions

Once the chips have been placed players take turns rolling a pair of dice. If a player has any chips on the dock whose number matches the sum of the dice, one chip is removed from that dock (the boat gets to cross the river), regardless of who rolled the dice. The game continues until one of the players has removed all of his chips. The player to do so is the winner, though in some cases ties are possible.

The presence of dock number 1 is superfluous since a sum of one is impossible. The choice of twelve chips to start the game is presumably made to correspond to the number of docks, but is otherwise arbitrary. We will consider games and positions with any number of chips.

Once the starting positions are decided the game play itself amounts to a probability experiment in which two dice are rolled until one of two sets of outcomes is achieved. The only strategy involved is in deciding the initial placement of the chips. Our primary goal then is to find the best initial configuration for a game with n chips.

At first glance this does not seem to be an imposing problem. For a game this simple one might expect to find an equally simple rule of thumb for determining the best starting position. If one exists we regret to report that we have not found it!

Notation

We will represent chip positions in two different ways. Our primary method will be to use a grid showing the number of chips assigned to each dock. The grid corresponding to the opening configurations shown in FIGURE 1 looks like this:

A		1	1	1	2	2	2	1	1	1	
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B				1	3	4	3	1			

When discussing opposing chip configurations we will refer to the top position as configuration A and the bottom as configuration B. Alternatively we denote the bottom configuration $B = \{5, 6^3, 7^4, 8^3, 9\}$. If $p(A \text{ wins}) \geq p(B \text{ wins})$ we say that A *dominates* B, and write $A \to B$. We will consider a configuration C to be *optimal* if $C \to A$ for all other configurations A with a like number of chips.

When comparing the probabilities of winning for opposing configurations we use the function

$$g(A, B) = \big(p(A \text{ wins}),\ p(B \text{ wins}),\ p(\text{tie game})\big).$$

For positions A and B from FIGURE 1 we can find that $g(A, B) \approx (.54, .41, .05)$. We also consider the *expected duration* of a configuration C which we denote $ed(C)$ and define as follows: Roll dice until all of C's chips have been removed, counting the number of rolls it takes to do so. Let X represent this count. Then $ed(C)$ is the expected value of X. For B above we find $ed(B) \approx 34$.

While the functions g and ed return only rational numbers, these often involve a large number of digits in both numerator and denominator. We therefore give decimal approximations unless there is a good reason to do otherwise.

Why the game is interesting

Where do we begin looking for optimal opening configurations? Since the game is essentially a race to eliminate one's chips, it makes sense to look for configurations which minimize the number of rolls needed to do so. This brings the following two hypotheses to mind:

1. The best initial chip configuration will be proportional to the probability distribution of the dice sums, or as close to proportional as possible after rounding.
2. The best initial chip configuration will be the one with the lowest expected duration.

Perhaps the first hypothesis was assumed by the inventors of the game. If it is true, then through repeated game play the astute player would learn to choose an initial configuration that resembles the familiar histogram of the distribution of the dice sums. The second hypothesis also seems reasonable. After all, if we are having a race shouldn't we choose the runner whose times are generally fastest? It may seem as well that these two hypotheses are both describing the same initial configuration.

To explore these hypotheses we relied first on computer based simulations, then developed algorithms for performing exact probability and expected value computations. In doing so we were surprised to find that both of these hypotheses are generally false, and that they do not typically describe the same configuration. In fact there are values of n for which both are true, for which neither is true, and for which one but not the other is true.

To test the first hypothesis we did simulations on the computer for games with $n = 36$ chips. This allowed us to make chip configuration A exactly proportional to the distribution of the dice sums, with one chip on dock 2, two chips on dock 3, etc. Our goal was to find other configurations which might be superior to A, though we did not really expect to find any. However by stacking more chips on the middle docks 6, 7, and 8 we found configurations that beat A consistently. The best we found experimentally is shown in configuration B below.

A	1	2	3	4	5	6	5	4	3	2	1
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B		1	3	4	6	8	6	4	3	1	

One simulation of 10,000 games yielded the experimental result $g(A, B) \approx (.31, .45, .24)$, so B seems clearly superior. This casts significant doubt on our first hypothesis. Through simulations we also found that $ed(A) \approx 81$ while $ed(B) \approx 70$, so aligning the chips proportionally to the probability distribution of the dice sums did not seem to yield the configuration with the smallest expected duration.

We have tried to verify these results with exact calculations but the number of computations required is too great even for our computers. However there are much simpler examples for which exact calculations yield the same general conclusions.

Once we were able to perform exact computations we quickly found a counterexample to Hypothesis 2. Consider the configurations

$$A = \{5, 6, 7^2, 8, 9\} \quad \text{and} \quad B = \{4, 5, 6, 7, 8, 9\}.$$

Here A has the least expected duration of all six chip configurations with $ed(A) \approx 19.8$, while $ed(B) \approx 21.2$. However $g(A, B) \approx (.247, .248, .505)$, so by a very slim margin $B \to A$. It turns out that B dominates all other six chip configurations as well, so for $n = 6$ it is optimal, even though it does not have the least expected duration.

While even our computers are unable to compute probabilities for games with large numbers of chips (roughly 25 or more per player), we will show how to do so when fewer chips are involved. Before discussing computational methods though we offer a few more instructive examples to help give a better understanding of the game.

Interesting positions

The following are positions that can arise in the course of play. They are not necessarily starting positions, so the players may have different numbers of chips. Unless otherwise noted the probabilities given are from exact calculations which have been rounded.

1. Here is a simple example which illustrates an important aspect of the game.

A					1	1					
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B						2					

Since 7 is the number most likely to be rolled, at first glance B with two 7's may seem to have an advantage. However the only way B can win is if two 7's occur before a single 6 is rolled. The odds of rolling a 7 before a 6 are 6 : 5, so the probability of this happening is $(6/11)^2 \approx 0.30$. Thus A has a significant advantage. In fact if the given position is altered by removing A's chip on dock 7, the probabilities of winning for both A and B are unchanged. That is,

$$g(\{6, 7\}, \{7^2\}) = g(\{6\}, \{7^2\}) = \left(\frac{85}{121}, \frac{36}{121}, 0\right).$$

This example illustrates a more general fact.

Chip removal property If Player A has more chips on a particular dock than Player B, the probabilities of winning for both players are unchanged if all of B's chips are removed from that dock.

This result is of course true when the roles of A and B are reversed. To further illustrate this idea, the probability of winning is the same for A and B in each of the following positions.

A				1	2	3	3				
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B				2	2	2	2	1			

A					2	3	3				
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B				2	2			1			

In either case $g(A, B) \approx (.29, .57, .14)$.

2. The chip removal property does not apply when players have an equal number of chips on a given dock. In the following position each player has a chip on dock 2, so the game will end in a tie if all the other chips are removed first.

A	1					2					
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B	1			1	1						

Here $g(A, B) \approx (.177, .175, .648)$, so by a slight margin $A \rightarrow B$.

If the chip on dock 2 is removed for each player the possibility of a tie no longer exists. Now each sequence of rolls that would yield a tie in the previous position yields a victory to either $A^* = \{7^2\}$ or $B^* = \{5, 6\}$, but these additional victories are apportioned differently. We now have $g(A^*, B^*) \approx (.498, .502, 0)$, and $B^* \rightarrow A^*$. The removal of the chips on dock 2 has not only changed the probabilities of winning, it has reversed the dominance relation as well. This shows that it is not possible to 'simplify' a position by removing an equal number of chips for each player from a given dock. We will revisit this idea with a rather amazing example at the end of this article.

3. This example illustrates how detrimental a chip on dock 2 or 12 can be to one's chances of winning. In this position A is down to the last chip but it is on dock 2, while B has several chips left. Who is more likely to win the game?

A	1										
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B		1	1	1	1	1	1	1			

Here B has a very slight advantage, with $p(A, B) \approx (.499, .501, 0)$. More often than not, each of the numbers 3 through 9 will be rolled before a 2 is rolled even once.

4. The dominance relation is generally not transitive. As in our second example we let $A = \{2, 7^2\}$, $B = \{2, 5, 6\}$, and now let $C = \{2, 4, 7\}$. We know from before that $A \rightarrow B$. Additionally we have

$$g(B, C) \approx (.208, .166, .626), \quad g(A, C) \approx (.174, .181, .645).$$

Here $B \rightarrow C$, and $C \rightarrow A$, so transitivity can not be assumed. As we search for optimal starting configurations we can never know in advance whether a configuration exists which dominates all others. For a given value of n there may be no single best opening configuration.

Computational methods

The computations required to find probabilities and expected values for this game are not difficult to understand, but quickly become so voluminous that a computer is required to perform them. As is often the case we can choose either direct or recursive methods. We give examples of both.

The following result should be intuitively clear, and is used repeatedly. Its proof is left to the reader.

THEOREM. *Let E and F be mutually exclusive outcomes of an experiment. Independent trials of the experiment are repeated until either E or F occur. Then the probability that E occurs before F is $p(E)/(p(E) + p(F))$.*

For example if E is the event "a sum of 5 is rolled," and F is the event "a sum of 6 or a sum of 7 is rolled," the probability that 5 is rolled before either a 6 or 7 is $(4/36)/(4/36 + 11/36) = 4/15$.

Suppose now that we are rolling dice until a 5, 6, and 7 have each occured. These will occur in one of 3! different orders. The probability that they occur in the order 5-7-6 is

$$\begin{aligned} p(5\text{-}7\text{-}6) &= p(5 \text{ occurs before 7 or 6}) \cdot p(7 \text{ occurs before 6}) \\ &= \frac{4}{4+6+5} \cdot \frac{6}{6+5} \\ &= \frac{8}{55}. \end{aligned}$$

The probability that 7 precedes 6 is unchanged whether a 5 precedes them both or not, so these events are independent and the computation above is justified.

Using this idea we can compute $g(A, B)$ for any two configurations A and B. We simply find the probability corresponding to each sequence of outcomes that eliminate all chips for both, and add the results appropriately. If $A = \{5, 6\}$ and $B = \{6, 7\}$, then all chips will be removed after 5, 6, and 7 have each been rolled once. If 7 is the last of these rolled then A wins, if 5 is last then B wins, and if 6 is last the game is a tie. This gives

$$\begin{aligned} p(A \text{ wins}) &= p(5\text{-}6\text{-}7) + p(6\text{-}5\text{-}7) \\ &= \frac{4}{15} \cdot \frac{5}{11} + \frac{5}{15} \cdot \frac{4}{10} \\ &= \frac{14}{55}. \end{aligned}$$

Similar computations give $p(B \text{ wins}) = 19/45$ and $p(\text{tie game}) = 32/99$.

While this gives us a straightforward way to compute $g(A, B)$ it is a challenge to implement this technique on a computer for games with larger numbers of chips. An alternative method involves recursion. This ultimately produces the same computations but is easier to program. The first recursive step is as follows:

$$g(\{5, 6\}, \{6, 7\}) = \frac{4}{15} \cdot g(\{6\}, \{6, 7\}) + \frac{5}{15} \cdot g(\{5\}, \{7\}) + \frac{6}{15} \cdot g(\{5, 6\}, \{6\}).$$

The first term of the sum on the right represents the probability that 5 is rolled before either 6 or 7, multiplied by $g(\{6\}, \{6, 7\})$, the conditional probability vector for this game given that 5 is rolled first.

The recursion terminates when either A or B is empty, or when $A = B$ and the game ends in a tie. We find $g(\{6\}, \{6, 7\})$ as follows:

$$\begin{aligned} g(\{6\}, \{6, 7\}) &= \frac{5}{11} \cdot g(\{\,\}, \{7\}) + \frac{6}{11} \cdot g(\{6\}, \{6\}) \\ &= \frac{5}{11} \cdot (1, 0, 0) + \frac{6}{11} \cdot (0, 0, 1) \\ &= \left(\frac{5}{11}, 0, \frac{6}{11}\right). \end{aligned}$$

Similar computations yield

$$g(\{5\}, \{7\}) = \left(\frac{4}{10}, \frac{6}{10}, 0\right) \quad \text{and} \quad g(\{5, 6\}, \{6\}) = \left(0, \frac{5}{9}, \frac{4}{9}\right).$$

We can now conclude that

$$\begin{aligned} g(\{5, 6\}, \{6, 7\}) &= \frac{4}{15} \cdot \left(\frac{5}{11}, 0, \frac{6}{11}\right) + \frac{5}{15} \cdot \left(\frac{4}{10}, \frac{6}{10}, 0\right) + \frac{6}{15} \cdot \left(0, \frac{5}{9}, \frac{4}{9}\right) \\ &= \left(\frac{14}{55}, \frac{19}{45}, \frac{32}{99}\right). \end{aligned}$$

To express the general recurrence relation for g we adopt some additional notation. Let $p_i = p(\text{a sum of } i \text{ is rolled})$. We now define

$$d_i = \begin{cases} p_i & \text{if either player has one or more chips on dock } i, \\ 0 & \text{if there are no chips on dock } i. \end{cases}$$

If we now let $s = \sum_{i=2}^{12} d_i$, the probability that the next chip to be removed comes from Dock i is d_i/s. We can now define g as

$$g(A, B) = \begin{cases} (0, 0, 1) & \text{if } A = B, \\ (1, 0, 0) & \text{if } A = \{\}, \\ (0, 1, 0) & \text{if } B = \{\}, \\ \sum_{i=2}^{12} (d_i/s) \cdot g(A - i, B - i) & \text{otherwise.} \end{cases}$$

Here $A - i$ is understood to be the configuration that results when one of A's chips on dock i is removed. If A has no chips on dock i then $A - i = A$, and similarly for B.

Expected duration computations can also be performed in a variety of ways. We repeatedly use the following result which appears in many undergraduate probability texts [**3**].

THEOREM. *Let F be an outcome of an experiment that occurs with probability p. Independent trials are conducted until F occurs. Let X represent the trial number on which F first occurs. Then the random variable X has a geometric distribution and $E(X) = 1/p$.*

For example, if F consists of rolling a sum of 5, 6, or 7, then the average wait for this to occur is $36/15 = 2.4$ rolls of the dice.

Suppose now that we wish to compute $ed(\{5, 6, 7\})$. When there is no more than one chip on each dock we can borrow an elegant formula from the literature on "coupon collecting" problems [**4**].

If X is the number of independent trials required to achieve each of three mutually exclusive outcomes with probabilities p_1, p_2 and p_3, we have

$$E(X) = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \left(\frac{1}{p_1 + p_2} + \frac{1}{p_1 + p_3} + \frac{1}{p_2 + p_3}\right) + \frac{1}{p_1 + p_2 + p_3}.$$

Note the similarity of this formula and the inclusion/exclusion formula for counting elements of sets. By keeping this in mind one can find the formula if other numbers of chips are involved. Substituting the dice probabilities for sums of 5, 6, and 7 into this formula gives $ed(\{5, 6, 7\}) = 151/11$.

Unfortunately this formula is difficult to generalize to cases where more than one chip appears on a particular dock (i.e., to collecting several coupons of a particular type). Once again recursion offers the easiest way to perform the computations we need. This time the first recursive step looks like this:

$$ed(\{5, 6, 7\}) = \frac{36}{15} + \frac{4}{15} \cdot ed(\{6, 7\}) + \frac{5}{15} \cdot ed(\{5, 7\}) + \frac{6}{15} \cdot ed(\{5, 6\}).$$

The first term of the sum is the expected number of rolls until a 5, 6 or 7 occurs. The rest of the formula is simply a weighted average of the number of rolls required to eliminate the remaining chips. To continue this example we find

$$\begin{aligned} ed(\{6,7\}) &= \frac{36}{11} + \frac{5}{11} \cdot ed(\{7\}) + \frac{6}{11} \cdot ed(\{6\}) \\ &= \frac{36}{11} + \frac{5}{11} \cdot \frac{36}{6} + \frac{6}{11} \cdot \frac{36}{5} \\ &= \frac{546}{55}. \end{aligned}$$

Similar computations give

$$ed(\{5,7\}) = \frac{57}{5} \quad \text{and} \quad ed(\{5,6\}) = \frac{61}{5}.$$

Our finished computation becomes

$$\begin{aligned} ed(\{5,6,7\} &= \frac{36}{15} + \frac{4}{15} \cdot \frac{546}{55} + \frac{5}{15} \cdot \frac{47}{5} + \frac{6}{15} \cdot \frac{61}{5} \\ &= \frac{151}{11}, \end{aligned}$$

which agrees with our previous result.

Using the same notation we adopted for g, the general recurrence relation is

$$ed(C) = \begin{cases} 0 & \text{if } C = \{\,\}, \\ \frac{1}{s} + \sum_{i=2}^{12} \left(\frac{d_i}{s}\right) \cdot ed(C - i) & \text{otherwise.} \end{cases}$$

Again $C - i$ is configuration C with one chip on dock i removed.

While recursive formulas are elegant they are often impractical to use on a computer, as they quickly exhaust memory resources and waste time recalculating results already found. One way around this is to have the computer store the result of each new computation, then look up this stored result when it needs to be found again. Without utilizing this technique many of our computational results would have been impossible for us to attain.

Conclusions

The only reliable method we found for identifying optimal configurations is to use our function g to exhaustively compare probabilities. This is not as difficult as it sounds for small n, since many configurations can be dismissed out of hand.

For the case $n = 12$ (the game played in our middle schools) the optimal configuration is $A = \{4, 5^2, 6^2, 7^3, 8^2, 9, 10\}$. Since sums of five and nine are equally likely, the same configuration but with only one chip on dock 5 and two chips on dock 9 is equal in strength.

For each of the cases $1 \leq n \leq 12$ there is a unique optimal configuration, disregarding symmetries of the type mentioned above. Each is symmetric about dock 7 or almost symmetric (the removal of one chip would make the configuration symmetric), due to the symmetry of the distribution of dice sums. We have not found a value for n for which no optimal configuration exists.

There is also a unique configuration which minimizes expected duration for $1 \leq n \leq 12$. It is equal to the optimal configuration for $1 \leq n \leq 5$ and for $n = 11$ and $n = 12$. For $6 \leq n \leq 10$ this is not the case. However when configurations with minimal expected durations are not optimal there are only one or two configurations which dominate them.

Our experimental evidence, that a chip configuration proportional to the distribution of the dice sums is not necessarily optimal, is confirmed with exact computations. For a game with $n = 24$ chips, distribution A shown below is as close to proportional as possible. The number of chips on dock i is the product of 24 and p_i, rounded to the nearest integer. There are several configurations which dominate A. All of these have more than four chips on dock 7. One such configuration is B shown below.

A	1	1	2	3	3	4	3	3	2	1	1
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B		1	2	3	4	5	3	3	2	1	

Here $g(A, B) \approx (.16, .43, .41)$. Configurations with as many as seven chips on dock 7 also dominate A. Configuration B may be optimal for $n = 24$, but we have not checked this in detail. Calculations of this size are near the limit of our computers' capabilities. This is not surprising since the exact probabilities returned for $g(A, B)$ by *Mathematica* are rational numbers which, expressed in simplest terms, have nearly 200 digits in both numerator and denominator!

TABLE 1: Special configurations for $1 \leq n \leq 12$

n	Optimal Configuration	Configuration w/Minimal *ed*	Minimal *ed*
1	$\{7\}$	$\{7\}$	6.00
2	$\{6, 7\}$	$\{6, 7\}$	9.93
3	$\{6, 7, 8\}$	$\{6, 7, 8\}$	12.50
4	$\{5, 6, 7, 8\}$	$\{5, 6, 7, 8\}$	15.48
5	$\{5, 6, 7, 8, 9\}$	$\{5, 6, 7, 8, 9\}$	17.77
6	$\{4, 5, 6, 7, 8, 9\}$	$\{5, 6, 7^2, 8, 9\}$	19.76
7	$\{4, 5, 6, 7^2, 8, 9\}$	$\{5, 6^2, 7^2, 8, 9\}$	22.28
8	$\{4, 5, 6, 7^2, 8, 9, 10\}$	$\{5, 6^2, 7^2, 8^2, 9\}$	24.31
9	$\{4, 5, 6^2, 7^2, 8, 9, 10\}$	$\{5, 6^2, 7^3, 8^2, 9\}$	26.47
10	$\{4, 5, 6^2, 7^2, 8^2, 9, 10\}$	$\{4, 5, 6^2, 7^3, 8^2, 9\}$	28.27
11	$\{4, 5, 6^2, 7^3, 8^2, 9, 10\}$	$\{4, 5, 6^2, 7^3, 8^2, 9, 10\}$	29.87
12	$\{4, 5^2, 6^2, 7^3, 8^2, 9, 10\}$	$\{4, 5^2, 6^2, 7^3, 8^2, 9, 10\}$	31.92

It is apparent that placing any chips on docks 2 and 12 is poor strategy for games with $n \leq 36$. The obvious explanation is that there is a significant chance that these will be among the last chips remaining near the end of the game, which puts the player at a severe disadvantage as in Example 3 earlier. This raises the following question: What is the smallest value of n that has an optimal configuration with a chip on dock 2 or 12? A more difficult question is this: As $n \to \infty$, what proportion of the chips should be placed on each dock? In other words, is there a limiting *shape* of the optimal distribution? The same questions can also be asked concerning distributions which minimize expected duration. We do not know the answers to these questions.

Before closing we offer a final example, one which is challenging to grasp intuitively. Consider games between configurations of the form shown below.

A				k	1						
Dock	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
B				k		2					

When $k = 0$ we found earlier that $g(A, B) \approx (.70, .30, 0)$. When $k = 1$ a tie is possible and

$$g(A, B) = \left(\frac{304}{605}, \frac{1136}{5445}, \frac{13}{45}\right) \approx (.50, .21, .29).$$

At this point we might wonder whether B's position relative to A has been improved by adding the chip on Dock 5. Let us define the function $b(A, B)$ to represent the proportion of the games not ending in ties that are won by B. In this case

$$b(A, B) = \frac{1136/5445}{304/605 + 1136/5445} \approx .293.$$

This indicates that B's position relative to A is slightly worse after adding the chip on Dock 5.

As more chips on Dock 5 are added to each configuration, B's position continues to deteriorate slightly until $k = 6$. At this point $b(A, B) \approx .279$ as shown in FIGURE 2.

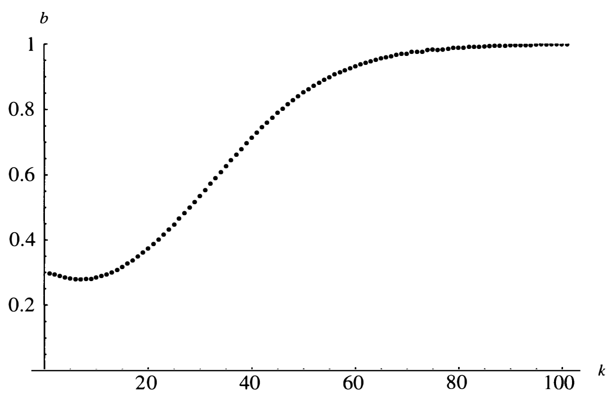

Figure 2 $b(A, B)$ for $0 \leq k \leq 100$

When the 7th chip is added to Dock 5 we find that B wins a slightly larger proportion of the games that are not tied. From this point on b increases rapidly. When $k = 28$ the dominance relation reverses and $B \to A$ for the first time. By the time $k = 100$, $b(A, B) \approx .998$. In fact we will find a closed formula for $b(A, B)$, and use it to prove that $b \to 1$ as $k \to \infty$.

Of course for large values of k the probability of a tie is very close to one. In fact when $k = 100$ this probability differs from 1 by less than $1/10^{35}$. What is difficult to understand is why, for those games in which all one hundred 5's are rolled before the 6 and two 7's are rolled, the overwhelming majority end with both 7's occuring before a single 6.

This behavior seems to occur regardless of the number of chips initially placed on Dock 7. For example, suppose we alter B's configuration by stacking fifteen chips on Dock 7 instead of just two, but leave A unchanged. When $k = 0$, fifteen 7's must be rolled before a single 6 is rolled in order for B to win. This occurs with probability $(6/11)^{15} \approx .0001$. As k increases initially things get even worse. For values of k between 100 and 200, $b(A, B)$ is within $1/10^9$ of 0. However as k increases, eventually b does as well. When $k = 530$, $b(A, B) \approx .52$, and when $k = 600$, $b(A, B) \approx .996$.

Returning to the case $A = \{5^k, 6\}$ and $B = \{5^k, 7^2\}$, we wish to show that $b(A, B) \to 1$ as $k \to \infty$. Recall that $b(A, B)$ gives the proportion of games not ending in ties that are won by B, so

$$b(A, B) = \frac{p(B \text{ wins})}{p(A \text{ wins}) + p(B \text{ wins})}.$$

We give the formulas first, then explain how we found them.

If we let $h(k) = p(B \text{ wins})$, then

$$h(k) = \left(\frac{4}{15}\right)^k \cdot \left[\left(\frac{5}{3}\right)^k - \frac{2}{11}k - \frac{85}{121}\right].$$

Similarly if $f(k) = p(A \text{ wins})$, then

$$f(k) = \left(\frac{4}{15}\right)^k \cdot \left[\left(\frac{3}{5}k + 1\right)\left(\frac{3}{2}\right)^k - \frac{12}{55}k - \frac{36}{121}\right].$$

Combining these and simplifying gives

$$b(A, B) = \frac{\left(\frac{5}{3}\right)^k - \frac{2}{11}k - \frac{85}{121}}{\left(\frac{5}{3}\right)^k + \left(\frac{3}{5}k + 1\right)\left(\frac{3}{2}\right)^k - \frac{2}{5}k - 1}.$$

To find the limit as $k \to \infty$ we multiply numerator and denominator by $(3/5)^k$. This gives

$$b(A, B) = \frac{1 - \frac{2}{11}k\left(\frac{3}{5}\right)^k - \frac{85}{121}\left(\frac{3}{5}\right)^k}{1 + \left(\frac{3}{5}k + 1\right)\left(\frac{9}{10}\right)^k - \frac{2}{5}k\left(\frac{3}{5}\right)^k - \left(\frac{3}{5}\right)^k}.$$

With L'Hospital's Rule we see that each non-constant term goes to 0 as $k \to \infty$, so $b \to 1$.

To find explicit formulas for f and h we use the nonrecursive technique explained earlier together with the summation formulas

$$\sum_{i=1}^{k} r^{(i-1)} = \frac{1 - r^k}{1 - r} \qquad \text{and} \qquad \sum_{i=1}^{k} i\, r^{(k+1-i)} = \frac{r}{1 - r}\left[k - r\left(\frac{1 - r^k}{1 - r}\right)\right].$$

The lesser known second formula can be derived from the first.

We begin by finding a formula for $h(k)$. Since B wins if and only if two 7's and k 5's occur before a single 6 is rolled, we find the probability associated with each such sequence of rolls and add them. There are three cases.

1. The sequence ends in $7 - 7 - 6$.

   Here all 5's occur first, so each 5 has probability $4/15$, while each 7 has probability $6/11$. Since there is only one way this can occur, the probability is

$$\left(\frac{4}{15}\right)^k \cdot \left(\frac{6}{11}\right)^2.$$

2. The sequence ends in $5 - 7 - 6$.

   Here the last 5 is preceded by one 7 and $(k - 1)$ 5's. These can occur in k different orders. The first 7 has probability $6/15$, the second $6/11$, and each 5 has probability $4/15$. So there are k such sequences, each with probability $(4/15)^k \cdot (6/15) \cdot (6/11)$, for a total of

$$k \cdot \left(\frac{4}{15}\right)^k \cdot \left(\frac{6}{15}\right) \cdot \left(\frac{6}{11}\right).$$

3. The sequence ends in $5 - 6$.

The last 5 is preceded by two 7's and $(k-1)$ 5's. Each 7 has probability $6/15$. Each 5 following the last 7 has probability $4/9$, while all other 5's have probability $4/15$. So if j 5's and one 7 precede the last 7, the probability for such a sequence is $(4/15)^j \cdot (6/15)^2 \cdot (4/9)^{k-j}$. There are a total of $(j+1)$ such sequences, with $0 \leq j \leq (k-1)$. Therefore the sum of all such sequences is

$$\sum_{i=1}^{k} i \cdot \left(\frac{6}{15}\right)^2 \cdot \left(\frac{4}{15}\right)^{i-1} \cdot \left(\frac{4}{9}\right)^{k+1-i}.$$

If we multiply by $(4/15)^k$ outside the summation and by $(15/4)^k$ inside, the sum is unchanged but can be rewritten as

$$\left(\frac{4}{15}\right)^k \cdot \left(\frac{6}{15}\right)^2 \cdot \sum_{i=1}^{k} i \cdot \left(\frac{15}{9}\right)^{k+1-i}.$$

Using the second summation formula, after simplifying this can be written

$$\left(\frac{4}{15}\right)^k \cdot \left[\left(\frac{5}{3}\right)^k - \frac{2}{5}k - 1\right].$$

The formula for $h(k)$ given earlier is the sum of these three expressions.

The formula for $f(k)$ is found similarly. Here A wins when a sequence containing k 5's, one 6, and two 7's ends with 7. This time there are a few more cases to consider. The cases and the sum of the probabilities resulting from each are given below without additional explanation.

1. The sequence ends in $6 - 7 - 7$. The probability is

$$\left(\frac{4}{15}\right)^k \cdot \frac{5}{11}.$$

2. The sequence ends in $7 - 6 - 7$. The probability is

$$\left(\frac{4}{15}\right)^k \cdot \frac{6}{11} \cdot \frac{5}{11}.$$

3. The sequence ends in $5 - 6 - 7$. This can happen in k ways. The probability is

$$k \cdot \left(\frac{4}{15}\right)^k \cdot \frac{6}{15} \cdot \frac{5}{11}.$$

4. The sequence ends in $5 - 7 - 7$. One or more 5's follow the 6. The total probability is

$$\frac{5}{15} \sum_{i=1}^{k} \left(\frac{4}{15}\right)^{i-1} \left(\frac{4}{10}\right)^{k+1-i} = \frac{5}{15} \cdot \left(\frac{4}{15}\right)^k \sum_{i=1}^{k} \left(\frac{15}{10}\right)^{k+1-i}$$
$$= \left(\frac{4}{15}\right)^k \left[-1 + \left(\frac{3}{2}\right)^k\right].$$

5. The sequence ends in $5-7$, and the 6 precedes the first 7. The total probability is

$$\frac{5}{15}\cdot\frac{6}{10}\cdot\sum_{i=1}^{k}(k+1-i)\left(\frac{4}{15}\right)^{i-1}\left(\frac{4}{10}\right)^{k+1-i},$$

which can be simplified as

$$\frac{5}{15}\cdot\frac{6}{10}\cdot\left(\frac{4}{15}\right)^{k}\sum_{i=1}^{k}(k+1-i)\left(\frac{15}{10}\right)^{k+1-i}$$
$$=\left(\frac{4}{15}\right)^{k}\left[\frac{6}{5}-\frac{6}{5}\cdot\left(\frac{3}{2}\right)^{k}+\frac{3}{5}k\cdot\left(\frac{3}{2}\right)^{k}\right].$$

6. The sequence ends in $5-7$, and the first 7 precedes the 6. The total probability is

$$\frac{6}{15}\cdot\frac{5}{15}\cdot\sum_{i=1}^{k}i\left(\frac{4}{15}\right)^{i-1}\left(\frac{4}{10}\right)^{k+1-i},$$

which can be simplified as

$$\frac{6}{15}\cdot\frac{5}{15}\cdot\left(\frac{4}{15}\right)^{k}\sum_{i=1}^{k}i\left(\frac{15}{10}\right)^{k+1-i}=\left(\frac{4}{15}\right)^{k}\left[-\frac{6}{5}+\frac{6}{5}\cdot\left(\frac{3}{2}\right)^{k}-\frac{2}{5}k\right].$$

The formula for $f(k)$ given earlier is the simplified sum of these six expressions. Finally $b(A,B)=h(k)/(f(k)+h(k))$.

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The cover image, called Parade Rest ©, was taken at Santee Basin on the grounds of the U.S. Naval Academy in 1991 in the fog. The picture won a prize in a photo contest run by Naval Proceedings and has been published in their magazine. The photo also has won ribbons in contests run by the Arundel Camera Club. The photographer, Howard Penn, is an active member of the MAA and Professor of Mathematics at the U.S. Naval Academy.

A Fresh Look at Peg Solitaire

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Peg solitaire is a one-person game that is over 300 years old; most people are familiar with the puzzle on the "standard 33-hole board" in FIGURE 1. When I first saw this game, what struck me was the unusual shape of the board. How was this strange cross-shaped board discovered and what is so special about it? While the history of the game is too fragmented to answer the question of the origin of this board, this paper will demonstrate that the special shape of the standard board can be derived from first principles. This board arises as a consequence of two very natural requirements: that of symmetry, and the ability to play from a board position with one peg missing to a single peg at the same location. We'll show that in a certain well-defined sense, the shape of this board is unique.

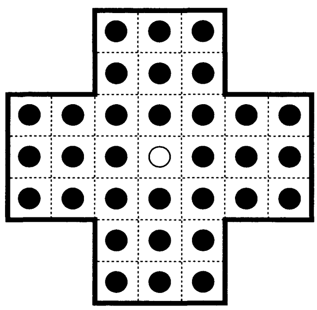

Figure 1 The standard 33-hole board

We refer to a board location as a **hole** because a physical board contains a hole or depression, in which the peg (or marble) sits. In all the diagrams, a hole with a peg is denoted by the symbol ●, while an empty hole is denoted by the symbol ○. The game begins with a peg in every hole except one, shown as the central hole in FIGURE 1. The player then jumps one peg over another into an empty hole on the board, removing the peg jumped over. No diagonal jumps are allowed, and the goal is to finish with one peg.

On the standard board, it is possible to start from the position in FIGURE 1 and finish with one peg in the center. Such a peg solitaire problem is called a **complement problem** because the starting and ending board positions are complements of one another (where every peg is replaced by a hole and vice versa). Note that all complement problems in this paper (by definition) start with one peg missing and finish with one peg.

In general, a board can be any region of holes on a square lattice. However the most aesthetically pleasing boards are those with some kind of symmetry.

Board symmetry

The highest degree of symmetry for a board (on a square lattice) is square symmetry. A **square-symmetric** board is unchanged by a reflection about either axis or either 45°

diagonal. Square-symmetric boards come in two varieties: **even** and **odd**, depending on whether their width is even or odd (or equivalently, the total number of holes T is even or odd). The standard 33-hole board is odd square-symmetric, and all such boards have a unique central hole. Even square-symmetric boards have a block of 4 central holes.

The pegs on odd boards can be divided into four categories: those that can reach the central hole, those that can jump over it vertically, those that can jump over it horizontally, and those that can neither reach it nor jump over it. Each peg remains in the same category for the entire game. On an even board, any peg can reach one and only one of the four central holes, and this also gives four categories of pegs. However the four jump patterns for an even board are simply reflections of one another. Because of this, in a general sense peg solitaire on even boards is less complex than on odd boards, and we expect that odd boards will produce more interesting and challenging problems.

We will use Cartesian coordinates to identify holes in a square-symmetric board, always placing the geometrical center of the board at the origin. On an odd board, the central hole is (0, 0), and all holes have integer coordinates. On an even board the four central holes are $(\pm 1/2, \pm 1/2)$, and all holes have half-integer coordinates. When we say one board is **smaller** than another, we always mean that the board has fewer holes.

A board is called **gapless** if, for any two holes on the board in the same column (or row), all the intervening holes are also on the board. This is equivalent to specifying that any horizontal or vertical line intersects the board either in a single interval, or not at all. Geometrically, saying a board is gapless is stronger than connectivity, but weaker than convexity. For example the standard 33-hole board is gapless (but not convex). Boards with interior voids or missing pieces along an edge are not gapless. Note that any jump must occur entirely on the board, and therefore if there is an interior void no jump is permitted into or over this void. For this reason boards that are not gapless can be cumbersome to play on, and we will consider only gapless boards, until the last section.

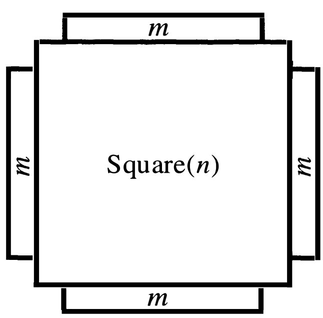

Figure 2 Augmenting a square-symmetric board.

The square board n holes on a side is certainly gapless and square-symmetric and will be called Square(n). What other gapless, square-symmetric boards are possible? Starting from Square(n), there is a geometrical technique to create a larger, square-symmetric board. We simply add a $1 \times m$ strip of holes symmetrically around all four sides as in FIGURE 2. To preserve square symmetry m must have the same parity as n, and we must have $m \leq n$ if the strips are not to overlap.

This process of adding strips of holes symmetrically to all four sides will be referred to as **augmenting** a board. Clearly we can repeat the process, adding another strip, and a whole (finite) sequence of strips of width m_i. In order that the final board be gapless, the integer sequence m_i must be *non-increasing*. When Square(n) is augmented by strips of width m_i, we'll denote the resulting board by Square(n) + (m_i). Let $\mathcal{B}$ be the set of all boards obtained by this construction.

$$\mathcal{B} = \{\text{Square}(n) + (m_i) \mid 0 < m_i \leq n \text{ non-increasing and } m_i \equiv n \pmod 2\}$$

PROPOSITION 1. *The set $\mathcal{B}$ contains* **all** *gapless, square-symmetric boards.*

Proof. By construction every board in $\mathcal{B}$ is gapless and square-symmetric. Is it possible that there is a gapless, square-symmetric board B that is not in $\mathcal{B}$? No, it isn't possible, because the gapless property ensures that the edge of the board must be formed from contiguous strips of holes, so we can remove them to obtain a smaller board that is still gapless and square-symmetric. We can continue this reduction inductively and it must terminate at a square board, so B $\in \mathcal{B}$. ∎

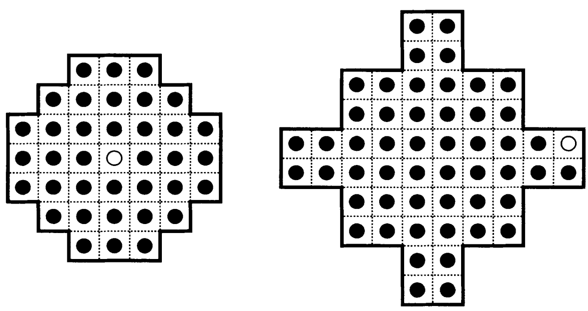

Figure 3 Sample elements of $\mathcal{B}$: (a) Square(5) + (3), known as the "French" board, (b) Square(6) + (2, 2).

FIGURE 3 shows two sample elements of $\mathcal{B}$. In this notation, the standard board of FIGURE 1 is Square(3) + (3, 3). Note that Square(n) + (m_i) has $T = n^2 + 4\sum m_i$ holes.

Null-class boards

Up until this section the rules of peg solitaire have not influenced the shape of the board, but we now determine properties that make for good peg solitaire boards. These stem from parity arguments along the diagonals [**1**], or alternatively the same theory can be derived from algebraic requirements [**2, 3**]. We use the former here because it is easier to understand the implications for square symmetry.

Consider two diagonal labelings of the holes of the board as shown in FIGURE 4 on square boards. Given a board position b, let $n_i(b)$ be the number of pegs in the holes marked i, and $t(b)$ be the total number of pegs on the board. A solitaire jump cannot change the parity of the differences $t - n_i$. This partitions the set of all possible board positions into **sixteen position classes** depending on the parity of the six integers $(t - n_i | i = 0, 1, \ldots, 5)$. Thus, all play is restricted to the position class of the starting position.

A **null-class** board is identified by the fact that b and the complement of b always lie in the same position class. In particular this must be true of the full and empty boards. We'll use the notation T and N_i for $t(b)$ and $n_i(b)$ when b is the full board.

T is the total number of holes in the board and N_i is the number of holes labeled i in FIGURE 4. The empty board lies in the position class where all six parities are even. Therefore a null-class board is one for which the six numbers $T - N_i$ are all even, or equivalently, all six N_i have the same parity (all odd or all even).

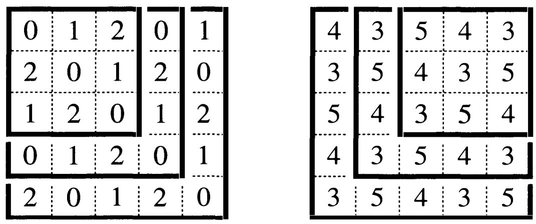

Figure 4 The labeling of holes on Square(n) where $n = 3$, 4 or 5.

For Square(3), we can see that all six $N_i = 3$, therefore this board is null-class. For $n = 4$ or 5, there is always an extra "0," or $N_0 = N_1 + 1$, and these boards are not null-class. In general, Square(n) is null-class if and only if n is a multiple of 3.

More interesting is the fact that *the process of augmenting a square board does not alter whether it is null class or not.* Why is this the case? If the augmentation process adds the hole (x_h, y_h), then it also adds the hole (y_h, x_h) reflected across the diagonal line $x = y$. The process never adds holes along the diagonal $x = y$, which ensures that $x_h \neq y_h$, so the holes are distinct. Because the parity labeling of FIGURE 4a is symmetric about the diagonal $x = y$, the two holes (x_h, y_h) and (y_h, x_h) are labeled the same, so *holes are always added in pairs with the same parity labels.* Therefore the parity of N_i does not change when the board is augmented. This completes a proof of the following proposition.

PROPOSITION 2. *Square(n) + (m_i) $\in \mathcal{B}$ is null-class if and only if n is a multiple of* 3.

Universal solvability

Why is null-class so important? Only on a null-class board can a board position and its complement be in the same position class. Therefore *a complement problem can only be solvable on a null-class board.* For this reason, null-class boards are the most interesting peg solitaire boards.

By Proposition 2, we know that the 37-hole "French" board of FIGURE 3a is not null-class and therefore no complement problem is solvable. In fact, the starting position for the central or (0, 0) complement problem is in the position class of the empty board, and cannot be reduced to a single peg, anywhere. The impossibility of solving a central vacancy to one peg is shared by **all** elements of $\mathcal{B}$ for which n is not a multiple of 3.

Just because a board is null-class, however, does not imply that any complement problem is solvable. In general we must investigate the particular board more fully to answer this question. We will call a board **universally solvable** if the complement problem is solvable at every board location.

The goal of the remainder of this paper is to determine which elements of $\mathcal{B}$ are universally solvable. This would appear an ambitious goal, because the task is not easy even for the standard 33-hole board (which is universally solvable). Nonetheless, we shall see that significant progress can be made. Null-class is a necessary condition for complement problem solvability, so we now concentrate on boards for which n is a multiple of 3.

PROPOSITION 3. *The* (0, 0) *complement problem is unsolvable on Square*(3) + (1, 1, ..., 1) *or Square*(3) + (3, 1, 1, ..., 1). *Here the sequence of consecutive 1's can have any length from zero to any positive integer.*

Proof. For boards of the first type, the (0, 0) complement problem is clearly unsolvable, because there is no way to remove the peg at (1, 1). For the boards of the second type, we use the resource count, or Pagoda Function shown in FIGURE 5. This is a real valued function of board position that (by construction) cannot increase during play. To calculate the value of this resource count for a particular board position, one sums the numbers where a peg is present. The reader should verify that no jump can increase the value of this resource count.

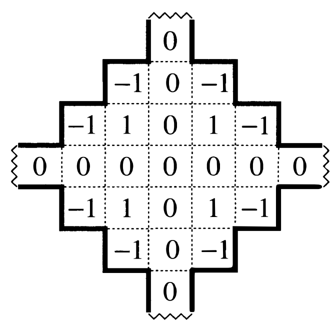

Figure 5 A resource count on Square(3) + (3, 1, 1, ..., 1).

For the central complement problem, this resource count begins at −4 and ends at 0; since solitaire jumps cannot increase the value of a resource count, it is impossible to reach the final state. In fact the same argument gives a much stronger result: no matter which peg is removed at the start, it is impossible to finish with fewer than 3 pegs. ■

THEOREM 1. *The standard* 33-*hole board Square*(3) + (3, 3) *is the smallest square-symmetric, gapless board that is universally solvable.*

Proof. This is immediate from Propositions 2 and 3, because the only null-class members of $\mathcal{B}$ that are smaller than the standard 33-hole board are those covered by Proposition 3. It is well known that the standard 33-hole board is universally solvable [**1, 2**]. ■

We can also identify the next largest universally solvable element of $\mathcal{B}$, the 36-hole board Square(6). This board is less interesting than the standard board due to its simpler geometry and the fact that it is even square-symmetric. Many other universally solvable boards can also be created by augmenting this board, such as Square(6) + (m_i), where (m_i) = (2), (2, 2), (4), (4, 2) or (6). We can show this by finding solutions to all complement problems.

Experienced peg solitaire players know that on the standard 33-hole board, the most difficult complement problem to solve is the (3, 0) complement (or symmetric equivalents). To obtain further intuition about larger boards, let us consider Wiegleb's board,

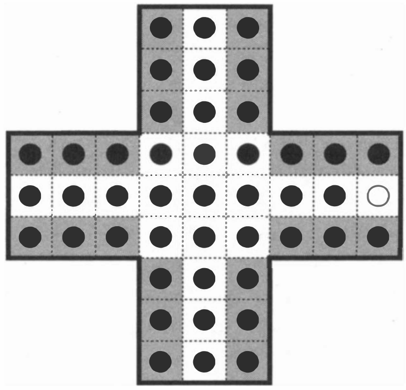

Figure 6 The board Square(3) + (3, 3, 3), "Wiegleb's Board." The significance of the shading will be explained in the proof of Theorem 3.

shown in FIGURE 6. This board was first introduced by J. Wiegleb in 1779 [**5**], but has since been relatively ignored.

Beasley [**1**, p. 200] states that all complement problems on Wiegleb's board are solvable *except* for the (4, 0) complement problem, with starting position shown in FIGURE 6 (or symmetric equivalents). The difficulty of solving the (4, 0) complement on Wiegleb's board is in fact a problem seen in all elements of $\mathcal{B}$: the most difficult complement problem to solve *begins from the center of the tip of the "arm."* Another example is the complement problem with starting position shown in FIGURE 3b, this problem is solvable but is the most difficult to solve on this board.

This suggests a useful generalization: we isolate the rightmost 3×3 section of Wiegleb's board (called "the needle" in the next section), and try to understand why the complement problem starting at the tip is difficult. The rest of the board (left of this 3×3 section) is not as important, and we can even allow it to be arbitrary. To solve the tip complement problem we must remove most of the pegs in the needle, but somehow build a trail of pegs to facilitate the final jumps back into the tip.

Boards with needles

Here we consider the general situation where a board of arbitrary shape has a $j \times m$ rectangular "needle" in the right half-plane $x > 0$, as in FIGURE 7b. The board in this

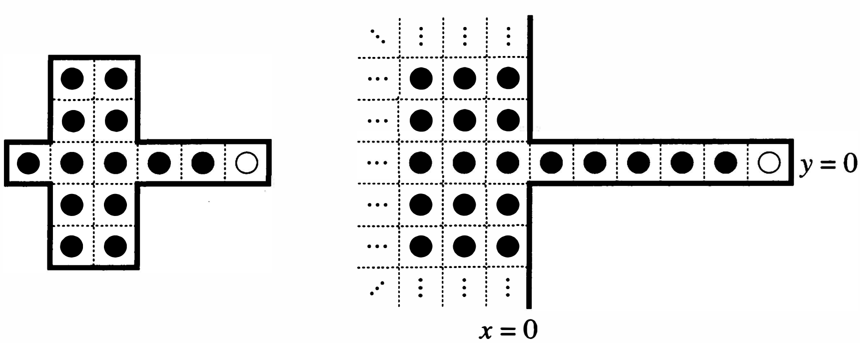

Figure 7 (a) A board containing a 1×3 needle. (b) A 1×6 needle attached to an arbitrary board.

section is not assumed to have any symmetry. Note that we have moved the coordinate origin to the base of the needle.

Let us first consider the needle of width $j = 1$. We want to answer the question: can we find a board containing a $1 \times m$ needle such that the complement problem at the tip of the needle is solvable? FIGURE 7a provides an example for the case $m = 3$. The reader should find the jumps for a solution to the tip complement problem; it will help to understand the problem and how to solve longer needles. The first two jumps are forced, after that you will find yourself doing a lot of rightward jumps to try to get a peg back to the end of the needle. The 1×4 and 1×5 needles are more difficult and require successively larger boards. Can we always solve longer needles by making the board larger and larger? No, as we will soon prove, the tip complement problem on a 1×6 needle is always unsolvable, no matter what board it is attached to. Although the right-half of the board is 1-dimensional, the left half is arbitrary, so this is not true 1D peg solitaire [**4**].

This problem is closely related to the "solitaire army" problem, a peg solitaire problem played on an infinite board [**1, 2**]. The solitaire army problem begins from a similar board position as FIGURE 7b, with pegs filling the entire left half-plane $x \leq 0$, and the goal is to jump a peg as far to the right as possible. The surprising result [**1, 2**] is that it is impossible to sent a scout (or peg) out 5 holes, no matter how many pegs are used. This result has been generalized to n-dimensions and diagonal jumps [**6, 7**], as well as other starting configurations [**8**].

Although similar to the solitaire army problem, our tip complement problem differs in several respects. Most significantly, there are pegs in the right half-plane at the start. More subtly, we cannot make any jump which is off the board, such as a rightward jump over (0, 1) in FIGURE 7b. Nonetheless, a similar technique is used to prove the following theorem.

THEOREM 2. *On any board with a $j \times m$ needle, for $j = 1, 2$ or 3 and $m > 5$, the tip or $(m, 0)$ complement problem is unsolvable.*

Proof. Consider the case $j = 1$ and the 1×6 needle of FIGURE 7b. We'll prove that the (6, 0) complement problem can't be solved (note that any longer needle can be considered a special case). To accomplish this, we use the resource count of FIGURE 8 (for the moment, ignore the values that are off the board). Let σ be the positive root of $x^2 + x - 1$, i.e. $\sigma = \frac{1}{2}(\sqrt{5} - 1) \approx .618$. σ is the reciprocal of the classical golden ratio. By construction $\sigma^2 + \sigma = 1$, and therefore

$$\sigma^i + \sigma^{i-1} = \sigma^{i-2} \qquad i \in \mathbb{Z} \tag{1}$$

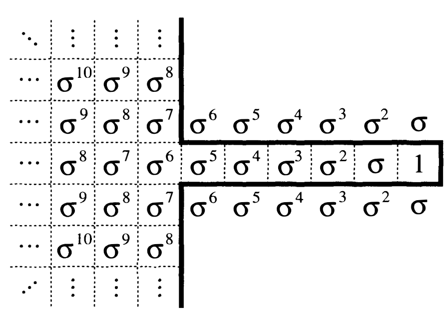

Figure 8 The resource count for the needle boards

It is this property that makes the pattern in FIGURE 8 a valid resource count, i.e., no jump can increase its total. In fact, rightward jumps lose nothing by (1), and the only jumps that reduce the total are

1. Leftward jumps, which lose an amount twice the hole jumped over
2. Vertical jumps away from $y = 0$, which lose an amount twice the hole jumped over
3. Vertical jumps over $y = 0$, which lose an amount equal to the hole jumped over

We can also express powers of σ by the formula

$$\sigma^i = (-1)^i [F_{i-1} - F_i \sigma] \qquad i \in \mathbb{Z} \tag{2}$$

Where F_i are the Fibonacci numbers, identified by $F_1 = F_2 = 1$, and $F_i = F_{i-2} + F_{i-1}$. Equation (2) can be proved by induction, and applies to all $i \in \mathbb{Z}$ if we extend the Fibonacci numbers by defining $F_0 = 0$, $F_{-i} = (-1)^{i+1} F_i$.

Now let us compute the total resource count in FIGURE 8 over the starting position in FIGURE 7b. First, we have the useful summing formula

$$\sum_{i=a}^{b} \sigma^i = \frac{\sigma^a - \sigma^{b+1}}{1 - \sigma} = \sigma^{a-2} - \sigma^{b-1} \qquad a \le b \tag{3}$$

The sum of all the values in the column $x = 0$, by (3) and (1), is

$$\sum_{i=6}^{\infty} \sigma^i + \sum_{i=7}^{\infty} \sigma^i = \sigma^4 + \sigma^5 = \sigma^3$$

Therefore the initial value of the resource count for the (6, 0) complement starting position is, using (3) and (2) is

$$\sum_{i=3}^{\infty} \sigma^i + \sum_{i=1}^{5} \sigma^i = \sigma + (\sigma^{-1} - \sigma^4) = 5\sigma - 1 \tag{4}$$

In reality the board is finite, and (4) provides an upper bound on the initial value of the resource count. If the initial value of the resource count (or an upper bound) minus the amount lost by any required jumps is less than the value of the final position, the problem is unsolvable. This computation will be called the **solvability criterion**: the problem is unsolvable if

$$\begin{bmatrix} \text{initial} \\ \text{resource} \\ \text{count} \end{bmatrix} - \begin{bmatrix} \text{amount lost} \\ \text{by required} \\ \text{jumps} \end{bmatrix} - \begin{bmatrix} \text{final} \\ \text{resource} \\ \text{count} \end{bmatrix} < 0$$

Note that after the first jump, there will be a peg at (6, 0) and this hole must be cleared before the final jump. The only possibility is a leftward jump over (5, 0), which loses 2σ in resource count. So the solvability criterion gives

$$[5\sigma - 1] - [2\sigma] - [1] = 3\sigma - 2 = -\sigma^4 < 0$$

Therefore the tip complement problem on the 1×6 needle is unsolvable.

For the case $j = 2$, we extend the 1-needle board of FIGURE 8 to include the holes at $(1 - 6, 1)$. The starting resource count value is given by (4), plus the amount added by the six additional holes. This amount is

$$\sum_{i=1}^{6} \sigma^i = \sigma^{-1} - \sigma^5 = 4 - 4\sigma \tag{5}$$

Combining (4) and (5) the starting value of the resource count is $3+\sigma$. After the first jump, we must clear the pegs at (6, 0) and (6, 1), which can only be accomplished by leftward jumps over (5, 0) and (5, 1), losing 2σ and $2\sigma^2$. It seems we must have additional leftward jumps to remove the pegs along $y=1$, but how can we be sure? This is answered neatly by the **exit theorems**, first stated by Beasley [**1**, p. 117], or see [**2**, p. 829]. One exit theorem states that any **region** of the board with at least 3 holes that starts out full but finishes empty must have at least two exits. An **exit** is any jump that removes a peg from the region and ends outside it. The first jump that removes a peg from the region must be an exit, and so must the last one.

Consider the region $R_1=(4-6,1)$. This region starts out full and finishes empty, so must have two exits, and these can only be the leftward jumps over (4, 1) or (3, 1), which each lose at least $2\sigma^4$. Likewise the region $R_2=(2-6,1)$ must have two exits, and these cannot be the same exits as for R_1. For R_2 we require two leftward jumps over (2, 1) or (1, 1), which each lose at least $2\sigma^6$. The solvability criterion therefore gives:

$$[3+\sigma]-[2\sigma+2\sigma^2+4\sigma^4+4\sigma^6]-[1]=45\sigma-28=-(3\sigma^6+\sigma^8)<0$$

So the (6, 0) complement problem on the 2×6 needle is unsolvable.

The final case is $j=3$; this adds another row of holes at $y=-1$. The initial resource count value, from (4) and (5), is $7-3\sigma$. The big change is that we now can clear (6, 0) with a vertical jump, let us suppose it is cleared by an upward jump. We then must have two leftward jumps over (5, 1), and as exits from R_1 and R_2 we can use two leftward jumps over (3, 1) and (1, 1) as before. In addition we require one leftward jump over $(4,-1)$, and for the regions $R_3=(3-5,-1)$ and $R_4=(1-5,-1)$ two exit jumps over $(2,-1)$ and $(0,-1)$. If we tally all this up, the solvability criterion yields

$$\begin{aligned}[7-3\sigma]-[1+4\sigma^2+4\sigma^4+4\sigma^6+2\sigma^3+4\sigma^5+4\sigma^7]-[1]&=19-31\sigma\\&=-(2\sigma^7+\sigma^5)<0\end{aligned}$$

In this case leftward jumps are not the only possible exits for the four regions. We can use two downward jumps over (4, 0) as exits for both R_1 and R_2, which lose $2\sigma^2$, and two upward jumps over (3, 0) as exits for both R_3 and R_4, which lose $2\sigma^3$. The solvability criterion then gives

$$[7-3\sigma]-[1+6\sigma^2+4\sigma^3]-[1]=3-5\sigma=-\sigma^5<0$$

We can also try clearing (6, 0) with a leftward jump, but the solvability condition is again negative. The (6, 0) complement on the 3×6 needle cannot be solved. ■

Is $m>5$ in Theorem 2 the best possible bound? FIGURE 9 shows a 56-hole board with a 1×5 needle where the tip complement problem is solvable (in fact this board is universally solvable). A 75-hole board with a 3×5 needle with solvable tip complement problem can be found in [**9**]. The 2×5 needle is the most difficult of the three—the smallest known board has 134 holes. Square-symmetric examples can be found in Square(15) + (1, 1, 1, 1, 1) and Square(15) + (3, 3, 3, 3, 3).

What about needles of width $m=4$ and beyond? Notice that any hole in a 4-needle has some horizontal and vertical jump into it. This extra freedom should allow us to find universally solvable examples that are as long as we like. For example, the 4×6 rectangular board by itself is universally solvable [**1**, p. 184]. It is not difficult to show that the $4\times m$ rectangular board is universally solvable for any $m\geq 6$ that is a multiple of 3 (using "packages and purges [**2**, p. 807]").

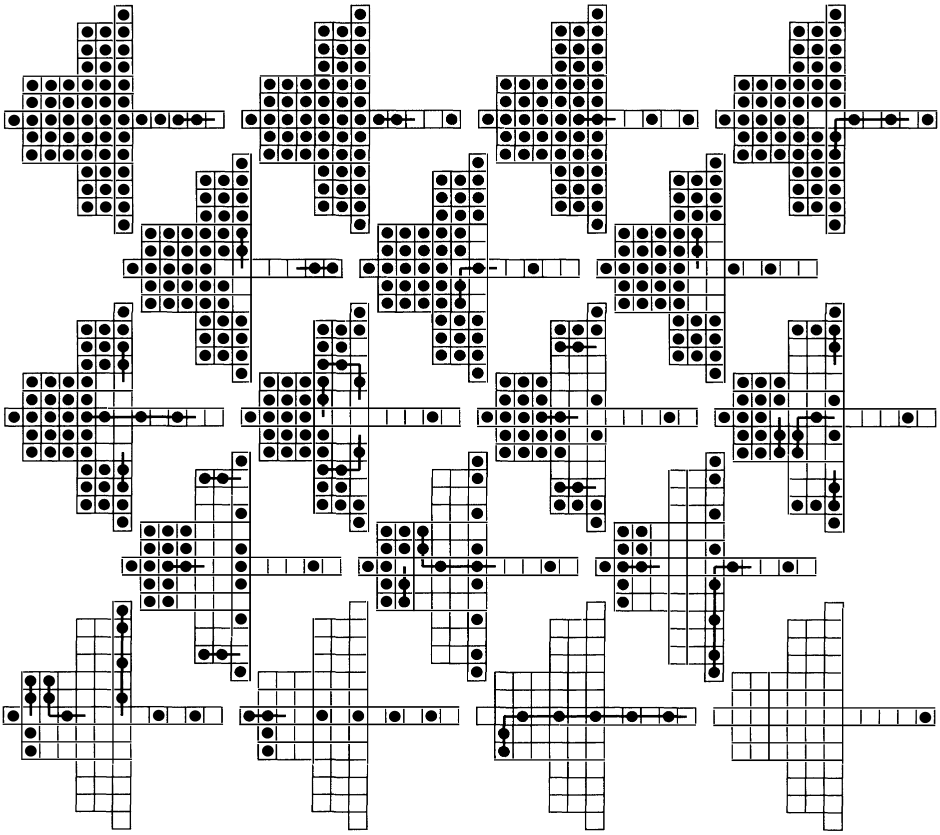

Figure 9 A board with a 1 × 5 needle with solvable tip complement problem.

Back to square symmetry

We can now use the results on needle boards to show that the standard board is even more remarkable.

THEOREM 3. *The standard* 33*-hole board is the only universally solvable board in* $\mathcal{B}$ *of the form Square*(3) + (m_i).

Proof. Here m_i must have the form $(3, 3, \ldots, 3, 1, 1, \ldots, 1)$; let n_3 be the number of 3's in m_i and n_1 the number of 1's. To prove this theorem, it suffices to show that the $(n_3 + n_1 + 1, 0)$ complement problem at the tip of the "arm" is unsolvable, except for the case $n_3 = 2$; $n_1 = 0$. Many cases are proved unsolvable by Proposition 3 or Theorem 2. In fact, Theorem 2 can be further generalized to show that the tip complement problem on any board with $n_3 + n_1 > 5$ is unsolvable. The proof uses exactly the same techniques as Theorem 2, and we omit it. This leaves a total of nine special cases: $n_3 = 2$, $n_1 = 1, 2, 3$; $n_3 = 3$, $n_1 = 0, 1, 2$; $n_3 = 4$, $n_1 = 0, 1$; $n_3 = 5$, $n_1 = 0$.

The first three boards can be handled using the resource count of FIGURE 10 (note the Fibonacci numbers along the x-axis). For the (4, 0) complement problem in FIGURE 10, this resource count begins at 45, and finishes at 21. But again the leftward jump to clear (4, 0) loses 26, so the solvability criterion gives $[45] - [26] - [21] = -2 < 0$.

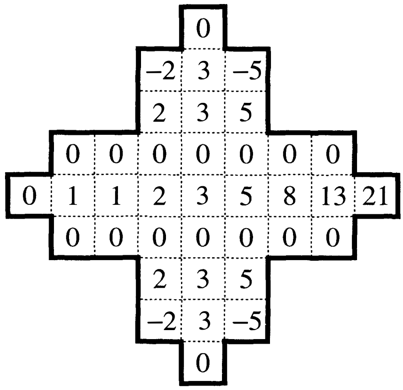

Figure 10 A resource count on Square(3) + (3, 3, 1)

This leaves six boards where the tip complement problem is difficult to prove unsolvable. For them, we use an integer programming (IP) model of the problem. We do not attempt to model the peg solitaire problem exactly (as in [**10**]), but allow any integer number of pegs in each hole, and a solitaire jump adds $(-1, -1, +1)$ to three consecutive holes. In this IP model, the order of the jumps is unimportant. For example, let's consider the (4, 0) complement problem on Wiegleb's board (FIGURE 6). On this board there are 108 geometrically possible jumps, and the number of each are our unknowns x_i. For each hole on the board, we have a linear equation which states that the starting number of pegs in this hole, minus the jumps that start from or jump over this hole, plus the jumps that end at this hole, equal the final number of pegs in this hole. This is a linear programming problem with 45 equations and 108 unknowns whose solution is restricted to non-negative integers, a standard problem for which computer solvers exist.

This IP model *is not* equivalent to the original peg solitaire problem, but it is solvable if the original problem is. Thus, if we can prove the IP model is unsolvable, it will prove the original problem unsolvable. Unfortunately, the (unmodified) IP model is solvable.

To complete the proof, we add to the IP model additional constraints that must be satisfied by the (4, 0) tip complement problem:

1. ≥ 2 rightward jumps into (4, 0) (the first and last jumps)
2. Exit requirements for each of the 8 shaded regions in FIGURE 6 (there are 5 possible exit jumps for each region)

When submitted to an integer programming solver (we recommend the free NEOS solver on the web [**11**]), the solver returns "integer infeasible." Similar computer proofs work for all 6 difficult boards. This is a rather subtle unsolvability, for if we take Wiegleb's board and remove the 3 holes at $x = -4$ the IP solver no longer reports that the (4, 0) complement is infeasible, and this 42-hole board can be shown to be universally solvable [**9**]. ■

A final remarkable fact comes immediately from Theorem 3, since we know (or can determine) that Square(9) is universally solvable.

COROLLARY. *Among odd square-symmetric, gapless boards, the standard* 33*-hole board is the only board with less than* 81 *holes that is universally solvable.*

Boards with gaps

Here we consider the role of the gapless assumption in the above analysis. If we require only that a board be square-symmetric and null-class, what strange and interesting boards may result? First, it is easy to see that any board that is square-symmetric is either null-class, or can be made null-class by removing or adding the central hole(s) (one hole for an odd board, all 4 for an even board). Because of this the number of null-class, square-symmetric boards is large. However the vast majority are uninteresting to play peg solitaire on, for the board may not be connected, or there may be a hole into which no jump is possible.

A computer search for the smallest universally solvable, square-symmetric boards came up with the two boards in FIGURE 11. The reader may enjoy finding solutions to all complement problems on these boards.

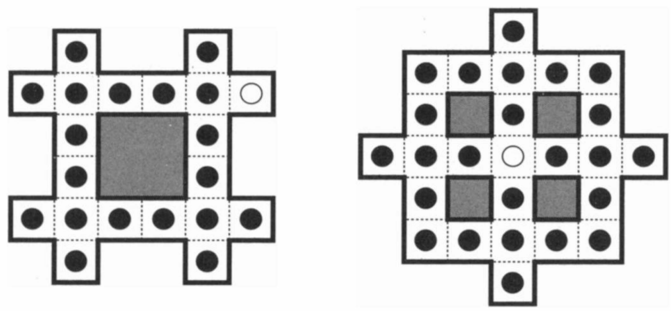

Figure 11 The smallest square-symmetric universally solvable boards (even and odd). Found by exhaustive computer search.

Conclusions

The concepts of null-class and symmetry provide a powerful combination for understanding peg solitaire boards. We have shown that the standard 33-hole board plays a special and unique role. It is the smallest gapless, square-symmetric board that is universally solvable. In fact it is even more special than this, because among gapless, odd square-symmetric boards, it is the only board with fewer than 81 holes that is universally solvable.

We should note that if we relax our symmetry requirements to rectangular symmetry, there are many universally solvable boards near the size of the standard board. For example, if we take the standard 33-hole board and remove the 6 holes at $y = \pm 3$, this 27-hole board is universally solvable. We can take Wiegleb's board and remove the 6 holes at $y = \pm 4$, this 39-hole board is also universally solvable, and the $(4, 0)$ complement problem has a *unique* solution, up to jump order and symmetry [**12**]. We can also play peg solitaire on a checkers board (allowing only diagonal jumps), this 32-hole board is universally solvable as well [**13**].

In this paper we have considered peg solitaire from a rather abstract perspective gained from years of exploration of the game, by hand and on a computer. We have given no actual solutions to problems, except for FIGURE 9. We hope the reader will be motivated to dust off a board (or find a computer version of the game) and try to solve the seven different complement problems on the standard board, and begin to explore problems on some of the other board shapes presented.

Acknowledgment. The author would like to thank John Beasley for helpful discussions and comments.

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Counting Cyclic Binary Strings

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Our objective is to illustrate by examples a particularly simple technique for enumerating some restricted subsets of $I_n = \{1, 2, 3, \ldots, n\}$. Some of these examples have appeared in the literature, enumerated there by more complicated methods; others are new and lead in some cases to combinatorial identities.

A subset $S \subseteq I_n$ is conveniently described by a sequence $\Delta = [\delta_1, \delta_2, \ldots, \delta_n]$ of 0's and 1's, called bits, where

$$\delta_i = \begin{cases} 1 & \text{iff } i \in S, \\ 0 & \text{iff } i \notin S. \end{cases}$$

We call Δ a binary n-bit linear string. If $i \in S$ and $i + 1 \in S$, $1 \leq i \leq n - 1$, we have an adjacent pair in S; in the corresponding Δ we have a pair of adjacent 1's, $\delta_i = \delta_{i+1} = 1$. In our examples we will consider 1 and n to be adjacent, for then interesting problems arise. This adjacency condition is seen when S is displayed in a circle, so we call such a subset cyclic. In the Δ corresponding to a cyclic subset of I_n we have a circular display of 0's and 1's with one of the n bits "capped" to indicate that it is δ_1. Since circular displays are difficult to typeset we display it linearly, enclosing it in round brackets to indicate that it should be visualized on a circle. We call it a cyclic n-bit string, briefly n-CS. For example

$$\begin{aligned} I_9 = \{1, 2, \ldots, 9\} \supseteq S = \{2, 3, 5, 7\} &\leftrightarrow \Delta = (\widehat{0}11010100) \\ &\equiv (0\widehat{0}1101010) \equiv (00\widehat{0}110101) \equiv \ldots \equiv (11010100\widehat{0}) \end{aligned}$$

Our method of enumerating cyclic subsets of I_n consists in making a sequence of constructions leading to the desired displays of 0's and 1's and counting the number of ways these constructions can be made. The method will become clear when we apply it to solve the problems which follow. Indeed, the solution to Problem 1 already makes the method transparent.

For convenience we take

$$\binom{n}{k} = \begin{cases} n!/k!(n-k)! & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\binom{n}{k} = 0$ when $k < 0$ or $n < 0$ or $0 \leq n < k$.

*Dedicated to my parents, Peggy and John McLeod.
†Dedicated to my grandchildren: Adam, Robert, Simon, Steven, Mina, and Sydney.

PROBLEM 1. Find the number $(n|k)$ of cyclic k-subsets of I_n with no two elements adjacent. Equivalently, for $0 \le k \le n$, $(n, k) \ne (0, 0)$, find the number $(n|k)$ of corresponding n-CS's: k 1's and $n-k$ 0's in a circle, one bit capped and every 1 followed clockwise by a 0.

Solution of Problem 1. It is well known ([**10**, p. 222, problem 2]) that

$$(n|k) = \begin{cases} \frac{n}{n-k}\binom{n-k}{k} & \text{if } 0 \le k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a particularly simple proof of this for $0 \le k < n$.
The formula states that:

(i) for $n \ge 1$, $(n|0) = 1$: the CS $(0, 0, \ldots, 0)$ is counted;
(ii) $(1|1) = 0$: there is no 0, so the 1 cannot be followed by a 0;
(iii) for $n \ge 2$, $(n|1) = n$: the single 1 is followed by $n - 1 \ge 1$ zeros, one of the n bits capped;
(iv) for $k > \frac{n}{2} > 0$, $(n|k) = 0$: there are k 1's and only $n - k < k$ zeros, so it is impossible for every 1 to be followed by a 0.

There remains the case $2 \le k \le \frac{n}{2}$. We build and count the appropriate circular displays of 0's and 1's as follows. Place $n-k$ 0's in a circle, creating $n-k$ indistinguishable boxes (the spaces between the 0's). Color one of the boxes, say blue; the boxes are now distinguishable. Choose k of these boxes, in $\binom{n-k}{k}$ ways, place a single 1 into each of the chosen boxes, "cap" one of the n entries, n ways to do this, erase the color and the $n\binom{n-k}{k}$ displays fall into sets, each of which has $n-k$ identical "uncolored" displays. Choose one display from each set and we have $\frac{n}{n-k}\binom{n-k}{k}$ displays, precisely those we want.

NOTE 1.1. By taking $(0|0) = 2$ and $(0|k) = 0$ if $k \ge 1$ (these have no combinatorial meaning) the numbers $(n|k)$, $k \ge 0$, $n \ge 0$, satisfy and are determined by the recurrence

$$\begin{aligned} &(n|k) = (n-1|k) + (n-2|k-1), && n \ge 2, \quad k \ge 1, \\ &(0|0) = 2,\ (n|0) = 1 \text{ for } n \ge 1, && (n|k) = 0 \text{ for } n = 0, 1,\ k \ge 1. \end{aligned} \tag{1}$$

The numbers $(n|k)$, $0 \le n \le 13$, $0 \le k \le 3$, are exhibited in the array below, with the initial values in boldface.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**2**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**0**	**0**	2	3	4	5	6	7	8	9	10	11	12	13
2	**0**	**0**	0	0	2	5	9	14	20	27	35	44	54	65
3	**0**	**0**	0	0	0	0	2	7	16	30	50	77	112	156

$(n|k)$, $0 \le n \le 13$, $0 \le k \le 3$

While an appropriate manipulation of binomial coefficients provides a proof of the recurrence (1) here is a combinatorial proof. Consider an n-CS counted in $(n|k)$, $n \ge 4$, $k \ge 2$. If the second 1 (counting clockwise from the capped bit) is followed by more than one 0, delete this 0 and there remains an $(n-1)$-CS counted in $(n-1|k)$ ways; if the second 1 is followed by exactly one zero, delete this 1 and the 0 and there remains a $(n-2)$-CS counted in $(n-2|k-1)$ ways.

NOTE 1.2. From recurrence (1) it is easy to deduce that the numbers

$$L_n = \sum_{k=0}(n|k),$$

which count the number of n-CS's with no two 1's adjacent, satisfy the recurrence

$$L_n = \begin{cases} 2, & n = 0, \\ 1, & n = 1, \\ L_{n-1} + L_{n-2}, & n \geq 2, \end{cases}$$

so they are the familiar Lucas numbers ([**3**], [**12**]):

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	...

NOTE 1.3. The numbers $(n|k)$ play an important role in the solution of the problème des ménages. In the reduced form of this problem, $2n$ chairs are placed in a circle around a table, n married men sit in alternate seats, and the problem is to find the number of ways that their wives can sit in the unoccupied chairs so that no wife sits next to her husband. An equivalent formulation is to ask for the number u_n of permutations $\{x_1, x_2, \ldots, x_n\}$ of I_n such that in every column of the $3 \times n$ array

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ n & 1 & 2 & \ldots & n-2 & n-1 \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \end{array}$$

the three integers are distinct. The number of such permutations is

$$u_n = \sum_{0 \leq k \leq n} (-1)^k (2n|k)(n-k)!, \qquad n \geq 2.$$

For a detailed analysis of this and related problems see ([**7**], [**9**], [**10**, p. 195]).

We will have occasion to use the following well-known result.

The number of distributions of n indistinguishable objects, $n \geq 1$, into k distinguishable boxes, $k \geq 1$, is

$$\binom{n+k-1}{k-1} = \binom{\#\text{objects} + \#\text{boxes} - 1}{\#\text{boxes} - 1}. \tag{2}$$

To see this, observe that each distribution can be seen as a linear display of n symbols $\bullet$ (bullets) and $k-1$ symbols $/$ (strokes):

$$\underbrace{\bullet\bullet\cdots\bullet}_{\text{box 1}} / \underbrace{\bullet\bullet\cdots\bullet}_{\text{box 2}} / \ldots / \underbrace{\bullet\bullet\cdots\bullet}_{\text{box k}} \qquad n \text{ bullets}, \quad k-1 \text{ strokes}$$

All such displays are created by lining up $n+k-1$ $\bullet$'s, choosing $k-1$ of them in $\binom{n+k-1}{k-1}$ ways, and changing the chosen $\bullet$'s into strokes.

The generalization is: for arbitrary but fixed $n \geq 0$, $k \geq 1$, $w \geq 0$, n indistinguishable objects can be distributed into k distinguishable boxes, each box receiving at least w objects, in

$$\binom{(n-kw)+k-1}{k-1} = \binom{n-k(w-1)-1}{k-1} \quad \text{ways.} \tag{3}$$

A bit is said to be *isolated* if it is not equal to either adjacent bit. Thus, in $(\cdots 1\,1\,0\,1 \cdots)$ the 0 is isolated and in $(\cdots 0\,1\,0\,0 \cdots)$ the 1 is isolated. Note that $(n|k)$ may also be described as the number of n-CS's with precisely k 1's, all isolated.

A substring of like bits not contained in a longer substring of like bits is called a *block*, e.g.,

$$(\ldots 1\underbrace{0\,0\,0\,0\,0\,0\,0}_{block}1\ldots) \qquad (\ldots 0\underbrace{1\,1\,1\,1\,1\,1\,1}_{block}0\ldots) \qquad (\ldots 0\underbrace{1\,\widehat{1}\,1\,1\,1}_{block}0\ldots)$$

The capped bit is in one of the blocks. Note that a bit which is in a block of length ≥ 2 is not isolated. That is, a bit is isolated precisely when it is a block of length 1.

Of course the n-CS's $(\widehat{0}\,0\ldots 0)$ and $(\widehat{1}\,1\ \ldots 1)$ have only one block. The n-CS's which contain both 0 bits and 1 bits all have an even number of blocks, the blocks of 0's alternating with blocks of 1's.

PROBLEM 2. Find the number of n-CS's having no isolated bits.

Solution of Problem 2. Equivalently, find the number of n-CS's, $n \geq 2$, whose blocks have various lengths ≥ 2. When $n \geq 2$ and $k = 0$ (resp. n) there is only one CS , namely $(\widehat{0}\,0\,0\ \ldots\ 0)$ (resp. $(\widehat{1}\,1\,1\ \ldots\ 1)$) neither of which has an isolated bit. In all other cases, $1 \leq k \leq n-1$, not all bits are alike, the n-CS has an even number of blocks, say 2ℓ, $\ell \geq 1$, and all blocks have length ≥ 2. Hence we wish to construct first the n-CS's with 2ℓ blocks (ℓ fixed and ≥ 1), all of various lengths ≥ 2. So, place 2ℓ strokes in a circle, creating 2ℓ indistinguishable boxes. Color one of the boxes; the boxes are now distinguishable. Distribute n symbols x (these will shortly be changed into 0's and 1's) into the boxes, at least two x's in each box; this can be done by (3) with $w = 2$ in

$$\binom{n-2\ell(2-1)-1}{2\ell-1} = \binom{n-2\ell-1}{2\ell-1}$$

ways. Cap one of the x's (in n ways). Now choose either 0 or 1 (2 choices). If your choice is 0 (resp. 1), replace all the x's in the box containing $\widehat{x}$ by 0's (resp. 1's), replace all the x's in the next box of x's by 1's (resp. 0's), continue replacing x's in succeeding boxes alternately by 0's (rep. 1's) and 1's (resp. 0's). At this point we have $2n\binom{n-2\ell-1}{2\ell-1}$ displays. Delete the stokes since they are no longer needed. Erase the color and the displays fall into sets of 2ℓ each which are identical. Choose one display from each set and we have: the number of n-CS's each with 2ℓ blocks all of which have length at least 2 is

$$\frac{2n}{2\ell}\binom{n-2\ell-1}{2\ell-1} = \frac{2n}{n-2\ell}\binom{n-2\ell}{2\ell} = 2(n|2\ell).$$

Summing over $\ell \geq 1$ and adding 2 for the n-CS's $(\widehat{0}\,0\,0\ \ldots\ 0)$, $(\widehat{1}\,1\,1\ \ldots\ 1)$, we have: the number of n-CS's with no isolated bits is

$$2 + \sum_{\ell \geq 1} 2(n|2\ell) = \sum_{\ell \geq 0} 2(n|2\ell) = \sum_{0 \leq \ell \leq n/2} \frac{2n}{n-2\ell}\binom{n-2\ell}{2\ell}, \quad n \geq 2, \tag{4}$$

whose values for $n = 2, 3, 4, \ldots, 14$ are

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$2\sum_{\ell \geq 0}(n\|2\ell)$	2	2	6	12	20	30	46	74	122	200	324	522	842

This enumeration was obtained by Agur, Fraenkl & Stein [**1**], who were motivated by a problem in genetic engineering (see [**12**]).

PROBLEM 3. Find the number of n-CS's, $n \geq 3$, which have no substring 0 0 0 nor 1 1 1 .

Solution of Problem 3. For $n \geq 3$ the two n-CS's $(\widehat{1}\ 1\ \dots\ 1)$, $(\hat{0}\ 0\ \dots\ 0)$ do not satisfy the condition. There remains to count the n-CS's, $n \geq 3$, with an even number of blocks each of which has length 1 or 2. We count these in subsets according to the number of blocks, say 2ℓ, $\ell \geq 1$, each having length 1 or 2, blocks of 0's alternating with blocks of 1's. We construct these as follows. Place 2ℓ strokes in a circle, creating 2ℓ indistinguishable boxes and color one of the boxes blue; the boxes are now distinguishable. We wish to distribute n symbols x—these will soon be changed to 0's and 1's—into the 2ℓ boxes, every box receiving either one or two x's. Consequently, precisely $4\ell - n$ of the boxes will each receive exactly one x, and precisely $n - 2\ell$ of the boxes will receive two x's. Hence, distribute the n x's into the 2ℓ boxes as follows. Put a single symbol x into each of the 2ℓ boxes, choose $n - 2\ell$ of the 2ℓ boxes, in $\binom{2\ell}{n-2\ell}$ ways, and put another symbol x into each of these chosen boxes, so that they contain two x's each. Now "cap" one of the n symbols x, in n ways, delete the color, and the $n\binom{2\ell}{n-2\ell}$ displays fall into sets, each containing 2ℓ identical uncolored displays. Choose one display from each set and we have $\frac{n}{2\ell}\binom{2\ell}{n-2\ell}$ displays. Now choose either 0 or 1 (2 choices). If your choice is 0 (resp. 1), replace all the x's in the box containing $\widehat{x}$ by 0's (resp. 1's), replace all the x's in the next box of x's by 1's (resp. 0's), continue replacing x's in succeeding boxes alternately by 0's (rep. 1's) and 1's (resp. 0's). Delete the strokes since they are no longer needed. At this point we have: the number of n-CS's with 2ℓ blocks ($\ell \geq 1$) all of length at most 2 is

$$\frac{n}{\ell}\binom{2\ell}{n-2\ell}, \quad n \geq 2,\ \ell \geq 1, \tag{5}$$

and the number of n-CS's, no 000 nor 111, is 4 if $n = 2$ and

$$\sum_{\ell \geq 1} \frac{n}{\ell}\binom{2\ell}{n-2\ell}, \qquad n \geq 3.$$

For $n = 3, 4, \dots, 8$ these values are 6, 6, 10, 20, 28, 46. This enumeration was obtained by Agur, Fraenkl & Stein [**1**], motivated by a problem in genetic engineering (see [**12**]). For the generalization of this problem to n-CS's each with all blocks having length $\leq w$ (w a fixed positive integer) see [**8**].

PROBLEM 4. For arbitrary but fixed $n \geq 1, k \geq 0, w \geq 1$, find the number $(n|k)_w$ of n-CS's, with precisely k 1's, each 1 followed by at least w 0's. (Note that $(n|k)_1 = (n|k)$.)

Solution of Problem 4. When $k = 0 < n$ an n-CS consists entirely of 0's and hence satisfies vacuously the condition "every 1 is followed by $\geq w$ 0's."

When $0 < \frac{n}{w+1} < k$ there are no "good" CS's.

There remains the case $1 \leq k \leq \frac{n}{w+1}$. We build and count the desired displays as follows. Place $n - kw$ 0's in a circle, creating $n - kw$ indistinguishable boxes. Color one of the boxes, say blue, so that the boxes are now distinguishable. Choose k of these boxes, in $\binom{n-kw}{k}$ ways, place a string $1\ 0\ 0\ \dots\ 0$, consisting of a 1 followed by $w - 1$ 0's, into each of the chosen boxes. Now "cap" one of the n entries, n ways to do this; erase the color and the $n\binom{n-kw}{k}$ displays fall into sets, each set containing $n - kw$ identical displays. Choose one display from each set and we have $\frac{n}{n-kw}\binom{n-kw}{k}$ displays, precisely those we want. We have, for $w \geq 0$,

$$(n|k)_w = \begin{cases} 1 & \text{if } k = 0 < n, \\ 0 & \text{if } 0 \leq n \leq w, k \geq 1, \\ \dfrac{n}{n-kw}\dbinom{n-kw}{k} & \text{otherwise.} \end{cases} \tag{6}$$

Taking $(0|0)_w = w+1$, the numbers $(n|k)_w$ are easily seen to satisfy, and are determined by,

$$(0|0)_w = w+1, \quad (n|0)_w = 1, \quad n \geq 1; \quad (n|k)_w = 0, \quad 1 \leq n < k(w+1);$$

$$(n|k)_w = (n-1|k)_w + (n-w-1|k-1)_w \quad , \quad n \geq k(w+1), \quad k \geq 1.$$

PROBLEM 5. For fixed $n \geq 2, r \geq 0$, find the number $(n|I(r))$ of n–CS's with exactly r isolated bits.

Solution of Problem 5. First, in the case where every bit is isolated, the only n-CS's are $(\widehat{0}1010\ldots01)$ and $(\widehat{1}010\ldots10)$, so

$$(n|I(n)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Note that if the n-CS consists entirely of 0's or entirely of 1's then it has no isolated bits. We move on to the case $n \geq 2, r \geq 1$, and construct these n-CS's in subsets according to the number of blocks, say 2ℓ, $\ell \geq 1$. Place 2ℓ strokes in a circle, creating 2ℓ indistinguishable boxes; color one of them. Choose r of the boxes, in $\binom{2\ell}{r}$ ways, and place a single symbol x into each of the chosen boxes. (x's will shortly become 0 's and 1 's.) Distribute $n-r$ x's into the $2\ell - r$ remaining (empty) boxes, at least 2 x's in each of them, in (by (3) with $w = 2$)

$$\binom{n-r-(2\ell-r)(2-1)-1}{2\ell-r-1} = \binom{n-2\ell-1}{2\ell-r-1}$$

ways. Cap one of the n x's (n ways) and so far we have

$$n\binom{2\ell}{r}\binom{n-2\ell-1}{2\ell-r-1}$$

circular displays of n x's in 2ℓ boxes with r boxes each containing a single x and $2\ell - r$ boxes each containing ≥ 2 x's, and one of the n x's capped. Erase the color and these displays fall into sets, each set containing 2ℓ identical displays. Choose one display from each set—$\frac{n}{2\ell}\binom{2\ell}{r}\binom{n-2\ell-1}{2\ell-r-1}$ displays—change the $\widehat{x}$ to a $\widehat{1}$ or $\widehat{0}$ (2 ways) and note that this choice determines all the other x's as 0 or 1. Now delete the strokes and we have constructed $\frac{2n}{2\ell}\binom{2\ell}{r}\binom{n-2\ell-1}{2\ell-r-1}$ displays, precisely those we want:

$$(n|I(r)) = \sum_{\ell \geq 1} \frac{n}{\ell}\binom{2\ell}{r}\binom{n-2\ell-1}{2\ell-r-1}, \qquad n \geq 3, r \geq 0.$$

The special case $r = 0$ is the number of n-CS's with no isolated bits:

$$(n|I(0)) = \sum_{\ell \geq 0} 2(n|2\ell), \qquad n \geq 2,$$

in agreement with (4).

PROBLEM 6. For arbitrary fixed $n \geq 2, k \geq 0, \ell \geq 1, w \geq 1$ find the number n-CS's each with precisely k 1's ($n-k$ 0's) and precisely 2ℓ blocks each having length at least w.

Solution of problem 6. Place 2ℓ strokes in a circle, creating 2ℓ indistinguishable boxes, and color one of the boxes. The boxes are now distinguishable: the colored box is B_1 and clockwise we have $B_2, B_3, \ldots, B_{2\ell}$. Distribute k 1's into the odd-numbered boxes, each box receiving at least w 1's; distribute $n-k$ 0's into the even numbered boxes, each box receiving at least w 0's. By (3) and (6) this can be done in

$$\binom{k-\ell(w-1)-1}{\ell-1}\binom{n-k-\ell(w-1)-1}{\ell-1} = \frac{\ell^2}{k(n-k)}(k|\ell)_{w-1}(n-k|\ell)_{w-1}$$

ways. Cap one of the n bits, delete the strokes and we have

$$\frac{n\ell^2}{k(n-k)}(k|\ell)_{w-1}(n-k|\ell)_{w-1}$$

displays. Erase the color, and we have: the number of n-CS's with precisely k 1's and 2ℓ blocks all of length at least w is

$$\frac{n\ell}{k(n-k)}(k|\ell)_{w-1}(n-k|\ell)_{w-1}\,. \tag{7}$$

When $w=1$ the condition "all blocks have length at least w" is satisfied by all n-CS's. In this case, the number of n-CS's with k 1's and 2ℓ blocks (an even number so $1<k<n$) is

$$\frac{n\ell}{k(n-k)}(k|\ell)_0(n-k|\ell)_0 = \frac{n\ell}{k(n-k)}\binom{k}{\ell}\binom{n-k}{\ell}.$$

Summing over ℓ we obtain: the number of n-CS's with k 1's and an even number of blocks is

$$\binom{n}{k} = \sum_{\ell>1}\frac{n\ell}{k(n-k)}\binom{k}{\ell}\binom{n-k}{\ell}, \qquad 1\le k<n.$$

([**2**], cf. identities 3.3 and 3.30).

The number of n-CS's with 2ℓ blocks ($\ell\ge 1$) all of length $\ge w$ ($w\ge 1$) is the sum of (7) over k and it is also $2(n|2\ell)_{w-1}$ (see Note 2.1). Hence we have the identity

$$2(n|2\ell)_{w-1} = \sum_{1\le k<n}\frac{n\ell}{k(n-k)}(k|\ell)_{w-1}(n-k|\ell)_{w-1},\ \ell\ge 1, n\ge 2, w\ge 1.$$

Replacing $w-1$ by $w\ge 0$ we have the identity

$$2(n|2\ell)_w = \sum_{1\le k<n}\frac{n\ell}{k(n-k)}(k|\ell)_w(n-k|\ell)_w, \qquad n\ge 2, \ell\ge 1, w\ge 0,$$

that is for $n\ge 2$, $\ell\ge 1$, $w\ge 0$,

$$\frac{2n}{n-2\ell w}\binom{n-2\ell w}{2\ell} = \sum_{1\le k<n}\frac{n\ell}{(k-\ell w)(n-k-\ell w)}\binom{k-\ell w}{\ell}\binom{n-k-\ell w}{\ell}.$$

The special case $w=0$ is the known identity ([**2**] identity 3.3)

$$2\binom{n}{2\ell} = \sum_{1\le k<n}\frac{n\ell}{k(n-k)}\binom{k}{\ell}\binom{n-k}{\ell}, \qquad 1\le \ell<n.$$

PROBLEM 7. For $1\le\ell$, $1\le k<n$, what is the number of n-CS's with exactly k 1's ($n-k$ 0's) and precisely 2ℓ blocks (blocks of 0's alternating with blocks of 1's), and no occurrence of 0 0 0 nor 1 1 1? In other words,how many n-Cs's have k 1's and each block has length 1 or 2?

Solution of Problem 7. Among the blocks of 1's, exactly $k-\ell$ have length two and $2\ell-k$ have length one, while among the blocks of 0's, exactly $n-k-\ell$ have length two and $2\ell-(n-k)$ have length one. Hence, to construct the desired displays, proceed as follows. Place 2ℓ strokes in a circle; color one of the boxes blue. B_1 is the blue box and $B_2, B_3, \dots B_{2\ell}$ follow clockwise. Choose $k-\ell$ of the odd-numbered boxes, put a pair 1 1 into these chosen boxes and a single 1 into the other $2\ell-k$ odd-numbered boxes; choose $n-k-\ell$ of the even-numbered boxes, put a pair 0 0 into these chosen boxes and a single 0 into the other $2\ell-n+k$ even-numbered boxes. Cap one of the n bits, erase the color, delete the strokes: the number of n-CS's with precisely k 1's and 2ℓ blocks all of length ≤ 2 is

$$\frac{n}{\ell}\binom{\ell}{k-\ell}\binom{\ell}{n-k-\ell}, \qquad 1 \leq \ell \leq k < n.$$

Summing over k, it follows from (5) that the number of n-CS's with 2ℓ blocks all of length ≤ 2 is

$$\frac{n}{\ell}\binom{2\ell}{n-2\ell} = \sum_{k\geq 1}\frac{n}{\ell}\binom{\ell}{k-\ell}\binom{\ell}{n-k-\ell}, \qquad 1 \leq \ell \leq k < n,$$

i.e., we have the identity

$$\binom{2\ell}{n-2\ell} = \sum_{k\geq 1}\binom{\ell}{k-\ell}\binom{\ell}{n-k-\ell}, \qquad 1 \leq \ell \leq k < n.$$

(Vandermonde's convolution ([**2**] identity 3.1; [**11**] p. 8))

PROBLEM 8. For $1 \leq k < n$, what is the number of n-CS's with exactly k 1's, none of them isolated?

Solution of Problem 8. We want to construct the circular displays of k 1's (and $n-k$ 0's), one of the n bits capped, and all blocks of 1's having length ≥ 2. We construct these displays in subsets according to the number of blocks of 1's, say $\ell \geq 1$. So, first display $n-k$ 0's in a circle and color one of the $n-k$ boxes so that the boxes (the spaces between the 0's) are now distinguishable. Choose ℓ of these boxes, $\binom{n-k}{\ell}$ ways, distribute the k 1's into these chosen boxes with $k \geq 2$ 1's into each box, by (3) with $w=2$, in $(\binom{k-\ell-1}{\ell-1}$ ways, cap one of the n bits (n ways) and delete the color. The $n\binom{n-k}{\ell}\binom{k-\ell-1}{\ell-1}$ displays fall into sets, each set containing $n-k$ identical displays. Choose one display from each set and we have

$$\frac{n}{n-k}\binom{n-k}{\ell}\binom{k-\ell-1}{\ell-1} = \frac{n\ell}{k(n-k)}\binom{n-k}{\ell}(k|\ell), \qquad \ell \geq 1, 1 \leq k < n.$$

Hence the number of n-CS's with exactly k 1's, none isolated, is

$$\sum_{\ell\geq 1}\frac{n\ell}{k(n-k)}\binom{n-k}{\ell}(k|\ell) = \frac{n}{k(n-k)}\sum_{\ell\geq 1}\ell\binom{n-k)}{\ell}(k|\ell), \quad 1 \leq k < n.$$

We hope that we have illuminated this elementary method of counting classes of n-CS's. The method can also be used to count classes of n-bit linear strings (see [**4**]). For related problems, see [**5**], [**6**].

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A Pointless Tale

In 1972–73 I was finishing my dissertation under the direction of Frank Harary at the Universsity of Michigan. Frank had a secretary who typed all his manuscripts and completed all his hand-drawn figures. One day when Frank happened to be out of town, Anne approached me with a perplexing problem.

"I finished typing this paper for Frank, but I can't find his drawing for FIGURE 1 anywhere. Do you think you can provide the needed drawing?"

I thought that I might be able to infer what graph is needed if I could read the paper and see the figure caption, provided I was familiar with the topic. Just a little hesitant, I asked, "Which paper is this?"

"It is by Harary and Ronald Read," she explained, "titled 'Is the Null-Graph a Pointless Concept?'"

Immediately I understood the issue. "The figure is fine, just leave a couple of inches blank above the caption."

"But there's nothing there," she insisted, "Why would anyone have a drawing of nothing?"

"Nothing is exactly what they want."

She turned away, muttering as she left, "You mathematicians are strange."

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A Sequence of Polynomials Related to the Evaluation of the Riemann Zeta Function

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The Riemann zeta function $\zeta(s)$ is defined for $s > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler was the first to evaluate the sum of the series for even values of s. Before 1740 he had obtained the values of $\zeta(2)$ and $\zeta(4)$, and in his *Introductio in analysin infinitorum* (1748) he explicitly gave the values for even s up to $s = 26$. Later he discovered the well-known formula valid for all even s in terms of the so-called Bernoulli numbers. (See [**2**] for an account of Euler's work.)

After Euler's original work many other methods have been given to compute $\zeta(2k)$ for integer k. Several of them have in common the use of moments of trigonometric functions (definite integrals of $x^k \cos nx$ or $x^k \sin nx$). We will modify that approach by defining a sequence of interpolating polynomials adapted in a natural way to the evaluation and express the sum $\zeta(2k)$ in terms of the values of the polynomials at a point. There is a recursive formula giving these values without using the polynomials themselves. Using the derivatives of these polynomials we obtain integral formulas for $\zeta(2k+1)$. This is achieved using only elementary calculus: trigonometric identities, integration by parts and the differentiability of some particular functions. Writing the Fourier series expansion of our sequence of polynomials we will recognize them as being essentially the Euler polynomials.

The basic ingredients

We will need two elementary trigonometric identities:

$$\frac{1}{2} + \cos\pi x + \cos 2\pi x + \cdots + \cos N\pi x = \frac{\sin(N+1/2)\pi x}{2\sin\pi x/2}, \tag{1}$$

$$\sin\pi x + \sin 3\pi x + \cdots + \sin(2N-1)\pi x = \frac{1-\cos 2N\pi x}{2\sin\pi x}, \tag{2}$$

and the following lemma.

LEMMA 1. *Let g be a function of class $\mathcal{C}^1$ on $[a, b]$. Then*

$$\lim_{\lambda\to\infty} \int_a^b g(x)\sin\lambda x\,dx = \lim_{\lambda\to\infty} \int_a^b g(x)\cos\lambda x\,dx = 0. \tag{3}$$

To prove the trigonometric identities multiply the left-hand side of (1) and (2) by the denominator of the right-hand side and use the trigonometric formulas for products of sines and cosines to get telescoping sums. To prove the lemma use integration by parts.

The reader familiar with the theory of Fourier series will recognize in (1) the Dirichlet kernel and in Lemma 1 the simplest form of the Riemann-Lebesgue lemma.

The evaluations will give the value of $\sum_{n=0}^{\infty}(2n+1)^{-s}$. This is enough to compute $\zeta(s)$ since we have

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s). \tag{4}$$

Evaluation of $\zeta(2k)$

An elementary way to evaluate $\zeta(2)$ is based on the following fact: the integral of $x\cos k\pi x$ on [0, 1] is $-2(k\pi)^{-2}$ for odd $k \geq 1$ and is 0 for even $k \geq 2$. Thus, multiplying both sides of (1) by x, integrating on [0, 1], passing to the limit, and using (3) and (4) we get $\zeta(2) = \pi^2/6$. This evaluation is essentially the same as the one given by E. L. Stark in [**10**], who used the second mean value theorem for integrals instead of (3).

Integrating $x^{2k-1}\cos(2n+1)\pi x$ instead of $x\cos(2n+1)\pi x$ lets us evaluate $\zeta(2k)$ recursively using the values of $\zeta(2l)$ for $l < k$. We will modify this approach and look for a polynomial of degree $2k-1$, denoted by p_{2k-1}, such that the integral of $p_{2k-1}(x)\cos(2n+1)\pi x$ on [0, 1] reduces to just one term, a constant (in n) multiple of $(2n+1)^{-2k}$. Integration by parts shows the conditions required on p_{2k-1}; we add the condition $p_{2k-1}(0) = 0$ to apply (3). Altogether we make the following definition.

DEFINITION 2. Let $p_1(x) = x$. For each integer $k \geq 2$ define p_{2k-1} as the unique polynomial of degree $2k-1$ with leading coefficient 1 such that $p_{2k-1}(0) = 0$ and

$$p_{2k-1}^{(2j-1)}(0) = p_{2k-1}^{(2j-1)}(1) = 0 \quad \text{for } j = 1, 2, ..., k-1. \tag{5}$$

Define also

$$p_{2k}(x) = \frac{p'_{2k+1}(x)}{2k+1} \quad \text{for } k \geq 1.$$

The polynomials p_k are uniquely defined. Since p_{2k-1} has leading coefficient 1 and vanishes at 0, there are $2k-2$ coefficients to be determined. Half of them are 0 (those corresponding to odd powers of x) because the derivatives of odd order vanish at 0. The remainder can be obtained one by one from the condition (5) at $x = 1$, starting with $j = k-1$. We will get an explicit formula later (equation (14)).

By integration by parts, the conditions on the derivatives of p_{2k-1} give exactly

$$\int_0^1 p_{2k-1}(x)\cos n\pi x\,dx = \frac{(-1)^k(2k-1)![1-(-1)^n]}{(n\pi)^{2k}}. \tag{6}$$

We are now ready to establish the first relationship between $\zeta(2k)$ and p_k. Let $N = 2M+1$. Multiply both sides of (1) by $p_{2k-1}(x)$, integrate on [0, 1] and use (6) to get

$$\frac{1}{2}\int_0^1 p_{2k-1}(x)\,dx + (-1)^k\frac{(2k-1)!}{\pi^{2k}}\sum_{n=0}^{M}\frac{2}{(2n+1)^{2k}}$$
$$= \int_0^1 \frac{p_{2k-1}(x)}{2\sin\pi x/2}\sin(2M+3/2)\pi x\,dx.$$

Taking the limit as M goes to infinity and applying (3) with

$$g(x) = \frac{p_{2k-1}(x)}{2\sin \pi x/2}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{(-1)^{k+1}\pi^{2k}}{4(2k-1)!} \int_0^1 p_{2k-1}(x)\,dx; \tag{7}$$

and using (4) we conclude that

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-2}\pi^{2k}}{(2^{2k}-1)(2k-1)!} \int_0^1 p_{2k-1}(x)\,dx. \tag{8}$$

Some properties of the polynomials p_k

The polynomial $p_{2k-1}(x)$ satisfies

$$p_{2k-1}(1) - p_{2k-1}(1-x) = p_{2k-1}(x). \tag{9}$$

To see this, note that the left-hand side fulfills all the requirements of Definition 2 and the interpolating polynomial is unique. From (9) we deduce several useful properties.

PROPOSITION 3. (i) $\displaystyle\int_0^1 p_{2k-1}(x)\,dx = \frac{1}{2}\,p_{2k-1}(1).$

(ii) $p_{2k-1}(1/2) = p_{2k-1}(1)/2.$

(iii) $p_{2k-1}^{(j)}(x) = (-1)^{j+1}p_{2k-1}^{(j)}(1-x),\ j = 1, \ldots, 2k-1.$

(iv) $p_{2k-1}^{(2j)}(1/2) = 0,\ j = 1, \ldots, k-1.$

(v) $p_{2k-1}(2) = 2.$

For (i) integrate both sides of (9) from 0 to 1 and make the change of variables $u = 1 - x$; (ii) and (iii) are easy consequences of (9); and (iv) follows from (iii). To prove (v) note that by Taylor's theorem,

$$p_{2k-1}(2) = p_{2k-1}(1) - p_{2k-1}(-1) = \sum_{j=0}^{2k-1} \frac{p_{2k-1}^{(j)}(0)}{j!}(1-(-1)^j),$$

and all the terms in the sum are zero except the one corresponding to $j = 2k - 1$.

Equations (iii) and (iv) also give information about the polynomials p_{2k}; for instance, $p'_{2k}(1/2) = 0$, a result that we will use later.

Using (i) in (8) we can state the following theorem.

THEOREM 4. *Let p_{2k-1} be the interpolating polynomial introduced in Definition* 2 *and denote $A_k = p_{2k-1}(1)$. Then*

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-3}}{(2^{2k}-1)(2k-1)!}\pi^{2k}A_k. \tag{10}$$

From the above computation of the coefficients of p_{2k-1} it should be clear that they are rational, so A_k is a rational number. As a consequence (10) shows that $\zeta(2k)$ *is a rational multiple of* π^{2k}.

Explicit calculations

For $k = 1$, we have $A_1 = 1$ and $\zeta(2) = \pi^2/6$. (This is the evaluation of $\zeta(2)$ mentioned above.) There are many proofs of this result: see, for instance, [**5**, Section 11.3] or the paper [**7**], and the references in both of them.

For higher values of k we use induction. The polynomial $p''_{2k-1}(x)$ has leading coefficient $(2k-1)(2k-2)$ and does not vanish at the origin; apart from these things it satisfies all the requirements of p_{2k-3}. Therefore

$$p''_{2k-1}(x) = (2k-1)(2k-2)(p_{2k-3}(x) - c) \tag{11}$$

for some constant c. Evaluating at $x = 1/2$ and using properties (iv) and (ii) in Proposition 3 we deduce $c = p_{2k-1}(1/2) = A_{k-1}/2$. Bringing this value to (11) we obtain

$$p''_{2k-1}(x) = (2k-1)(2k-2)(p_{2k-3}(x) - \frac{1}{2}A_{k-1}), \tag{12}$$

which yields the following rule: *to get* p_{2k-1} *from* p_{2k-3}, *integrate term by term twice, add* $-A_{k-1}x^2/4$ *and normalize the leading coefficient.*

From (12) and Proposition 3 we also obtain

$$p''_{2k-1}(0) = -p''_{2k-1}(1) = -\frac{(2k-1)(2k-2)}{2}A_{k-1},$$

and so by induction

$$p^{(2j)}_{2k-1}(0) = -p^{(2j)}_{2k-1}(1) = -\frac{(2k-1)!}{2(2k-2j-1)!}A_{k-j}. \tag{13}$$

This gives an explicit formula for $p_{2k-1}(x)$, namely

$$p_{2k-1}(x) = x^{2k-1} - \frac{1}{2}\sum_{j=1}^{k-1}\binom{2k-1}{2k-2j}A_j x^{2k-2j}. \tag{14}$$

Evaluating it at $x = 1$ gives a recurrence formula for A_k:

$$A_k = 1 - \frac{1}{2}\sum_{j=1}^{k-1}\binom{2k-1}{2k-2j}A_j. \tag{15}$$

Thus we can fill the following table.

k	p_{2k-1}	A_k	$\zeta(2k)$
1	x	1	$\frac{\pi^2}{6}$
2	$x^3 - \frac{3}{2}x^2$	$-\frac{1}{2}$	$\frac{\pi^4}{90}$
3	$x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2$	1	$\frac{\pi^6}{945}$
4	$x^7 - \frac{7}{2}x^6 + \frac{35}{4}x^4 - \frac{21}{2}x^2$	$-\frac{17}{4}$	$\frac{\pi^8}{9450}$
5	$x^9 - \frac{9}{2}x^8 + 21x^6 - 63x^4 + \frac{153}{2}x^2$	31	$\frac{\pi^{10}}{93555}$

Other recurrence formulas for A_k can be found by evaluating (14) at $x = 1/2$ and $x = 2$ and using Proposition 3 to replace $p_{2k-1}(1/2)$ and $p_{2k-1}(2)$. More interesting for us is the formula obtained by differentiating (14) and evaluating at $x = 1$; doing this after replacing $2k - 1$ by $2k + 1$ in (14) we obtain

$$\sum_{j=1}^{k} \binom{2k+1}{2j} j A_j = 2k + 1, \tag{16}$$

which can be rewritten as

$$\sum_{j=1}^{k} \binom{2k}{2j-1} A_j = 2.$$

In search of the old formula

The classical formula for $\zeta(2k)$ uses Bernoulli numbers of even order and is

$$\zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \pi^{2k} B_{2k}. \tag{17}$$

Bernoulli numbers can be obtained from the Taylor expansion of $x/(e^x - 1)$, namely

$$\frac{x}{e^x - 1} = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!},$$

or can be computed using the recurrence relation

$$\sum_{j=0}^{k} \binom{2k+1}{2j} B_{2j} = \frac{2k+1}{2} \quad (k \geq 1), \tag{18}$$

starting with $B_0 = 1$. To check that (10) is the same as (17) we need to show that

$$k A_k = 2(2^{2k} - 1) B_{2k}. \tag{19}$$

This will follow if we prove that both sides coincide for $k = 1$ and satisfy the same recurrence relation. Equality for $k = 1$ holds because $B_2 = 1/6$; on the other hand, according to Lemma 2 in [**1**, p. 429] the Bernoulli numbers satisfy

$$\sum_{j=0}^{k} \binom{2k+1}{2j} 2^{2j} B_{2j} = 2k + 1. \tag{20}$$

Subtracting (18) from (20) and multiplying by 2 we obtain for $2(2^{2j} - 1)B_{2j}$ the recurrence relation given by (16) for jA_j.

Apart from Euler's method, there are many other ways of evaluating $\zeta(2k)$; see, for instance, [**1**], [**3**], [**4**], [**6**] and [**9**]. The proofs in [**3**] and [**9**] also use sequences of polynomials.

Integral forms of $\zeta(2k + 1)$

It is well-known that no explicit evaluations of $\zeta(2k + 1)$ have yet been obtained. Instead, there are equivalent expressions given by other series or definite integrals. Here we present a formula obtained using the sequence p_{2k}.

Take (6) with $k+1$ instead of k and integrate by parts to deduce

$$\int_0^1 p_{2k}(x)\sin(2n+1)\pi x\,dx = \frac{(-1)^k(2k)!2}{[(2n+1)\pi]^{2k+1}}.$$

Multiply both sides of (2) by $p_{2k}(x)$, integrate from 0 to 1 and take the limit as N goes to infinity to get

$$\frac{(-1)^k(2k)!2}{\pi^{2k+1}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+1}} = \int_0^1 \frac{p_{2k}(x)}{2\sin\pi x}\,dx = \int_0^{1/2}\frac{p_{2k}(x)}{\sin\pi x}\,dx. \tag{21}$$

(The last equality is obtained by changing x into $1-x$ in the integral from $1/2$ to 1 and using $p_{2k}(x) = p_{2k}(1-x)$ from Proposition 3, (iv).) This yields the following theorem.

THEOREM 5. *Let $\{p_k\}$ be the sequence of polynomials introduced in Definition* 2. *Then*

$$\begin{aligned}\zeta(2k+1) &= \frac{(-1)^k 2^{2k}\pi^{2k+1}}{(2^{2k+1}-1)(2k)!}\int_0^{1/2}\frac{p_{2k}(x)}{\sin\pi x}\,dx\\ &= \frac{(-1)^k 2^{2k}\pi^{2k}}{(2^{2k+1}-1)(2k-1)!}\int_0^{1/2}\left(p_{2k-1}(x)-\frac{A_k}{2}\right)\log\cot\frac{\pi x}{2}\,dx.\end{aligned}$$

The second formula is obtained from the first one by integrating by parts and using (12), since

$$\frac{d}{dx}\log\cot\frac{\pi x}{2} = -\frac{\pi}{\sin\pi x}.$$

(For the evaluation at 0 remember that $p_{2k}(0) = 0$ and $\log\cot\pi x/2 \sim \log 1/x$.) For $k=1$ and $k=2$ we get

$$\zeta(3) = \frac{2\pi^3}{7}\int_0^{1/2}\frac{x-x^2}{\sin\pi x}\,dx = \frac{2\pi^2}{7}\int_0^{1/2}(1-2x)\log\cot\frac{\pi x}{2}\,dx,$$

and

$$\zeta(5) = \frac{2\pi^5}{93}\int_0^{1/2}\frac{x^4-2x^3+x}{\sin\pi x}\,dx = \frac{2\pi^4}{93}\int_0^{1/2}(4x^3-6x^2+1)\log\cot\frac{\pi x}{2}\,dx.$$

Using Fourier series to recognize our polynomials

For a function f defined on $[0, 1]$, its Fourier cosine series is

$$\frac{a_0}{2}+\sum_{j=1}^{\infty}a_j\cos j\pi x,$$

where

$$a_j = 2\int_0^1 f(x)\cos j\pi x\,dx.$$

This corresponds to the Fourier series of the (unique) even function of period 2 whose restriction to $[0, 1]$ is f. One of the simplest convergence theorems for Fourier series states that *if f is continuous and piecewise differentiable, then the Fourier series of f converges (uniformly) to f*. See, for instance, [**11**] for this and the results below on Fourier series.

The coefficients of the Fourier cosine series of $p_{2k-1}(x)$ are

$$a_0 = A_k, \quad a_{2n+2} = 0, \quad a_{2n+1} = \frac{(-1)^k 4(2k-1)!}{((2n+1)\pi)^{2k}} \; (n = 0, 1, 2, \ldots).$$

The value of a_0 is given by Proposition 3 (i); the other coefficients are in (6).

Using the convergence theorem we get the equality

$$p_{2k-1}(x) = \frac{A_k}{2} + \frac{(-1)^k 4(2k-1)!}{\pi^{2k}} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi x}{(2n+1)^{2k}}, \quad 0 \leq x \leq 1. \tag{22}$$

Particular values of x in (22) provide the sum of the corresponding numerical series. For instance, choosing $x = 0$ (or $x = 1$) in (22) leads to the evaluation of $\zeta(2k)$.

The derivative of the even extension of $p_{2k+1}(x)$ to $[-1, 1]$ is odd and coincides with $(2k+1)p_{2k}(x)$ (see Definition 2) on $[0, 1]$. The corresponding periodic function of period 2 is continuous and has piecewise continuous derivative. From the relation between the Fourier coefficients of a function and those of its derivative, and from the convergence theorem for Fourier series we deduce

$$p_{2k}(x) = \frac{(-1)^k 4(2k)!}{\pi^{2k+1}} \sum_{n=1}^{\infty} \frac{\sin(2n+1)\pi x}{(2n+1)^{2k+1}}, \quad 0 \leq x \leq 1. \tag{23}$$

Again, particular values of x in (23) allow us evaluate numerical series. An interesting case is $x = 1/2$: this yields the value of the so-called Dirichlet beta function, $\beta(s) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$, for odd s.

Comparing (22) with the Fourier series expansion of the Euler polynomials we see that $p_{2k-1}(x) - A_k/2 = E_{2k-1}$. The sequence $\{E_n\}$ of Euler polynomials is defined by

$$\frac{2e^{xt}}{1+e^t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

See [**8**, Chapter 20] for the definition and properties of the Euler polynomials including their Fourier series expansion. The property $E_n' = nE_{n-1}$ shows that E_{2k} coincides with p_{2k}.

This suggests that Euler polynomials could be defined as interpolating polynomials using some variant of Definition 2. Indeed, it would be enough to replace the condition $p_{2k-1}(0) = 0$ (introduced because it was suited to our aim of using Lemma 3) with $p_{2k-1}(0) = -p_{2k-1}(1)$ or, alternatively, with $p_{2k-1}(1/2) = 0$.

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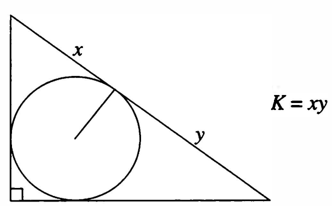
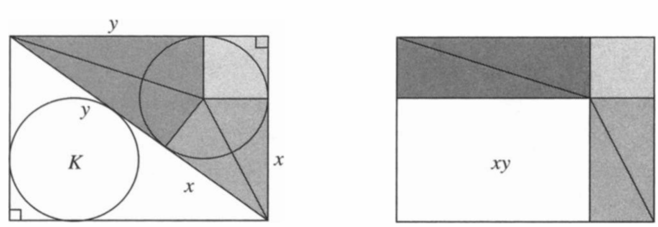
Proof Without Words: The Area of a Right Triangle

THEOREM. *The area K of a right triangle is equal to the product of the lengths of the segments of the hypotenuse determined by the point of tangency of the inscribed circle.*



Proof.



$K = xy$

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Brussels Sprouts and Cloves

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Introduction

Sprouts is a well-known 2-person pen-and-paper game; see **[20]**, **[18]**, **[9]**, **[15]**, **[12**, Chap. 37], and the *World Game of Sprouts Association* Web site **[19]**. One starts with an agreed number of dots on the page. A *move* in this game is played by connecting two (possibly equal) dots by a simple curve (carefully avoiding any other dot) and introducing a new dot on the middle of the curve. There are only two rules:

1. no two curves can cross,
2. no more than 3 curves can issue from any dot.

The players take turns to move; the loser is the first person who has no possible move. FIGURE 1 shows a game with one initial dot; here the second player has won.

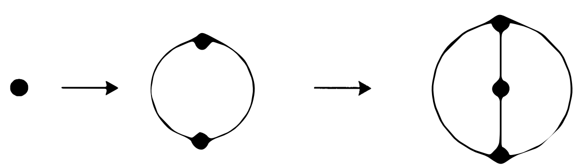

Figure 1

Introduced by Michael S. Paterson and John H. Conway in 1967, Sprouts has interesting combinatorial and topological aspects **[7, 17]**. In particular, there is the daunting *Sprouts conjecture*: for a game with n initial dots, there is a winning strategy for the first player if and only if n is congruent to 3, 4 or 5 modulo 6. This conjecture has been verified only for small values of n, by (computer) exhaustion **[1]**.

There is a variation of Sprouts called Brussels Sprouts; here one begins with crosses rather than dots, and with each move one introduces a new cross by marking a bar across the middle of the curve. FIGURE 2 shows a game with one initial cross.

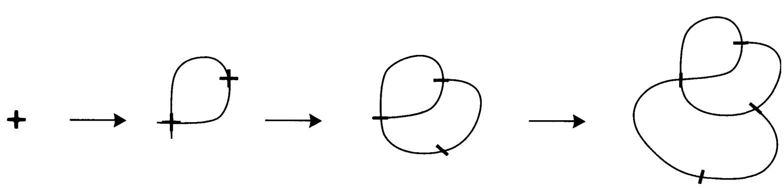

Figure 2

Due to Conway, Brussels Sprouts is something of a mathematical joke. A simple Euler characteristic calculation shows that there is no strategy involved in the game; for Brussels Sprouts played with n initial crosses, the first player wins if and only if n is odd, regardless of how the game is played [**9, 3**]. In fact, the same is true if one plays the game on any orientable surface, rather than the plane. It is perhaps less well known that on non-orientable surfaces, Brussels Sprouts is more interesting; here it is a game of strategy. Nevertheless, the game is amenable to analysis. One has:

THEOREM 1. *For a game of Brussels Sprouts on a compact surface M, without boundary, of Euler characteristic χ, played with m initial crosses, there is a winning strategy for the second player if and only if m and χ are both even.*

Theorem 1 is stated and key elements of a proof are outlined in [**13**]. It is also stated in [**10**]. As far as we are aware, a detailed proof of the theorem hasn't appeared in the literature.

The purpose of this paper is to introduce a generalization of Brussels Sprouts, which we call Cloves, and to state and prove Theorem 1 in this more general context. Before defining Cloves, let us describe the sort of game that we will be able to treat with Cloves; in doing so, the choice of name "Cloves" will become apparent. Consider a number of dots on the plane with a non-zero number of "free arms" connected to each dot, as in FIGURE 3.

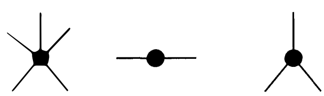

Figure 3

As in Brussels Sprouts, a move is made by connecting two free arms with a curve and marking a bar across the middle of the curve, thus adding two new free arms, one on each side of the curve. Notice that like Brussels Sprouts, such a game has the key feature that as the game is played, the number of free arms remains constant; in each move one uses two arms and adds two new free arms. The game of Cloves that we will now define is equivalent to the game we just described, but its presentation is quite different.

DEFINITION 1. A game of *Cloves* begins with a compact surface M with b boundary curves and on each boundary curve there is a non-zero number of dots. A *move* is played as follows:

(a) connect two distinct dots by a simple curve γ and remove the two dots,

(b) cut M along γ,

(c) introduce two new dots, one on each of the curves resulting from the splitting of γ.

The game finishes, as in Sprouts, when a player can't make a move.

FIGURE 4 shows a game of Cloves, starting with a single disc with three dots. The surfaces in FIGURE 4 have been drawn to emphasize the cuts that have been made, and the dotted lines shows the moves that are about to be made. The same game is more conveniently drawn as in FIGURE 5. FIGURE 6 shows a game of Cloves starting with an annulus with two dots.

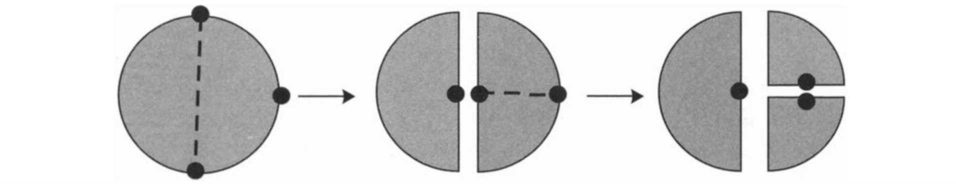

Figure 4

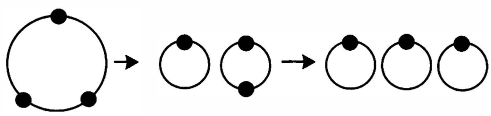

Figure 5

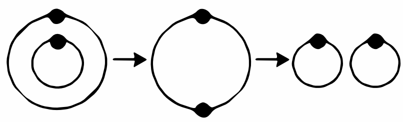

Figure 6

In what way is Cloves a generalization of Brussels Sprouts? Consider a game of Brussels Sprouts, starting with a collection of crosses on some surface M. For each cross, we do the following: rub out the cross, remove a small disc from the surface where the cross was located, and put 4 dots around the boundary curve. In Brussels Sprouts, we draw a curve γ between two crosses; since the subsequent curves cannot cross γ, the curve γ is effectively part of the boundary of the playing surface. In Cloves, this is formalized by splitting the surface along γ. For example, FIGURE 7 shows a game of Brussels Sprouts (thought of as being played on a large sphere), while FIGURE 8 shows the corresponding game of Cloves.

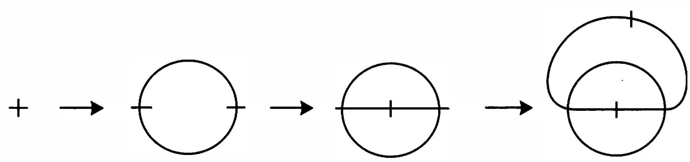

Figure 7

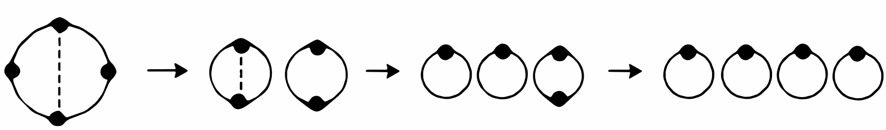

Figure 8

Fundamental theory of surfaces

Before continuing with Cloves, we need to recall some of the fundamental results of surface theory. The two basic constructions on a surface are the *handle* and the *cross-*

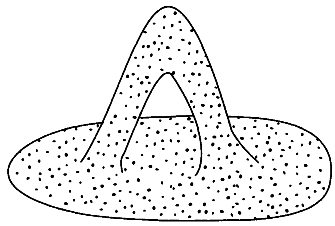

Figure 9

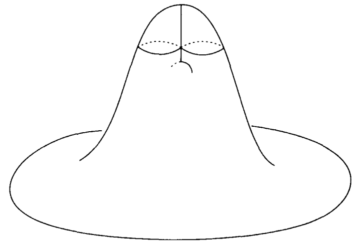

Figure 10

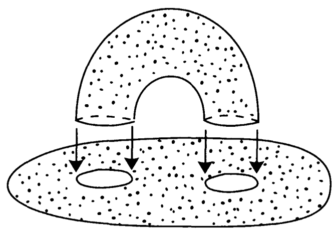

Figure 11

cap which are shown in FIGURE 9 and FIGURE 10, respectively. Both surfaces have a single boundary curve. The handle is easy to understand; it is obviously orientable and can be constructed by removing two small discs from a bigger disc, and attaching a cylinder to the small holes, as in FIGURE 11. In order to avoid 3 dimensional drawings, we will depict handles as in FIGURE 12. For example, FIGURE 13 shows a closed curve that passes under and over the handle. Notice that if you attach a disc to the boundary of a handle, you get a torus.

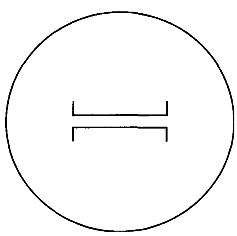

Figure 12

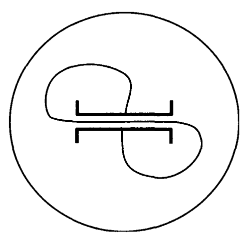

Figure 13

Cross-caps are non-orientable and are harder to visualize; see [**14, 2**]. Nevertheless, they are easy to understand since they are equivalent topologically to Möbius strips. Recall that a Möbius strip is obtained from an annulus, by cutting and reglueing it with a twist, as shown in FIGURES 14 and 15. If you attach a disc to the boundary of a cross-cap (or Möbius strip), you get a projective plane [**5**]. We represent cross-caps as in FIGURE 16. The convention here is that as a curve passes across the cross-cap symbol, the orientation is reversed, as shown in FIGURE 17.

Recall that the Euler characteristic χ of a triangulated surface M is the alternating sum $V - E + F$ of the number of its vertices, edges and faces, respectively. (See [**8**, Ch. 5] and [**16**, Part 1, Ch.1] for information about the Euler characteristic and Euler's formula.) The following theorem summarizes some of the fundamental results of surface theory; see [**11, 5, 10**].

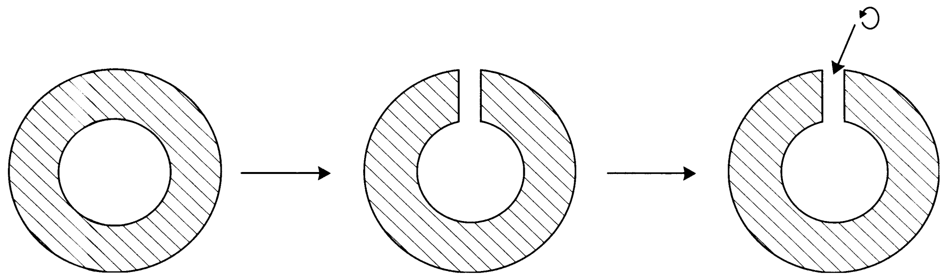

Figure 14

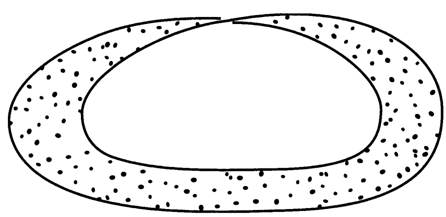

Figure 15

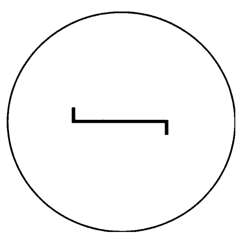

Figure 16

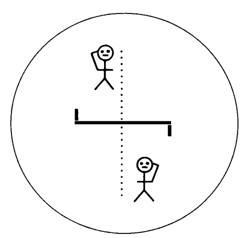

Figure 17

THEOREM 2.

(a) *Every compact connected orientable surface M with boundary is homeomorphic to a disc with a number, h say, of holes and a number, g say, of handles. The Euler characteristic of such a surface is $\chi = 1 - h - 2g$ and M has $h + 1$ boundary curves.*

(b) *Every compact connected non-orientable surface M with boundary is homeomorphic to a disc with h holes and k cross-caps. The Euler characteristic of such a surface is $\chi = 1 - h - k$ and M has $h + 1$ boundary curves.*

(c) *Any two compact connected orientable (resp. non-orientable) surfaces that have the same number of boundary curves and the same Euler characteristic are homeomorphic.*

Remark 1. Even the first part of this theorem is non trivial; FIGURE 18 shows one of the consequences. Perhaps the most striking and non-intuitive part of the above theorem is its corollary: a disc with three cross-caps is homeomorphic to a disc with one cross-cap and one handle; see [**11**].

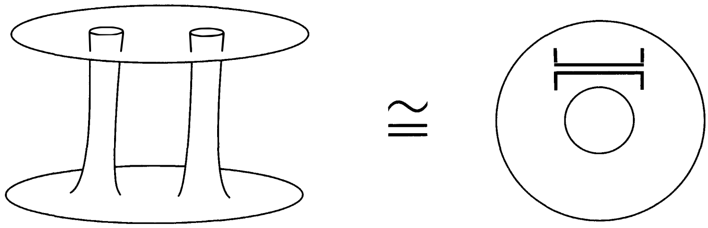

Figure 18

Remark 2. Notice that part (b) of Theorem 2 fails unless M is connected. The problem here is that in general M may have some connected components that are orientable and some components that aren't. Only on the non-orientable components can each handle be replaced by a pair of cross-caps. Notice that if M has c connected components and b boundary curves, then by applying the above theorem to each component separately, one obtains $\chi(M) \leq 2c - b$.

Notation. In what follows, we will write $\chi(M), b(M), c(M)$ etc., if there is any possible confusion as to the surface M.

Cutting up surfaces

As Cloves are played by cutting surfaces along curves, let us briefly recall the effect of this construction on the topology of the surface. The general situation is that we have a connected compact surface M with b boundary components, and Euler characteristic $\chi(M)$, and we cut M along a simple curve γ that joins distinct points on the boundary of M. Consider the resulting surface M'.

LEMMA 1. $\chi(M') = \chi(M) + 1$.

Idea of proof. It suffices to notice that each move adds one to the alternating sum $V - E + F$ of vertices, edges and faces in any triangulation; see FIGURE 19, where E has been reduced by one, while V and F are unchanged. ■

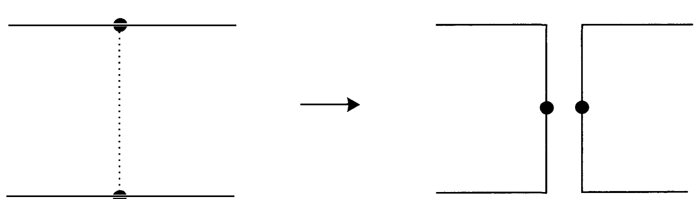

Figure 19

If the end points of γ lie on distinct boundary components, M' has one fewer boundary curves than M, and is still connected; see FIGURE 20.

If the end points of γ lie on the same boundary component, there are two possibilities. Let $\hat{\gamma}$ denote the loop obtained by composing γ with a path back around the boundary of M (there are two choices for $\hat{\gamma}$ depending on which return path one takes along the boundary). If $\hat{\gamma}$ is an orientation preserving loop, the resulting surface M'

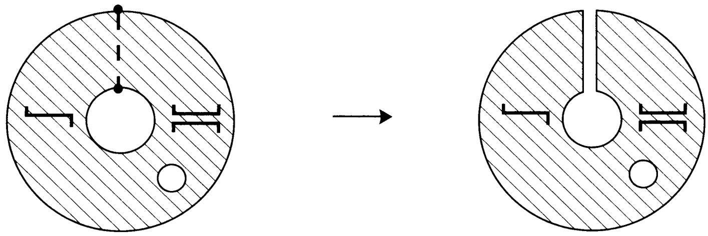

Figure 20

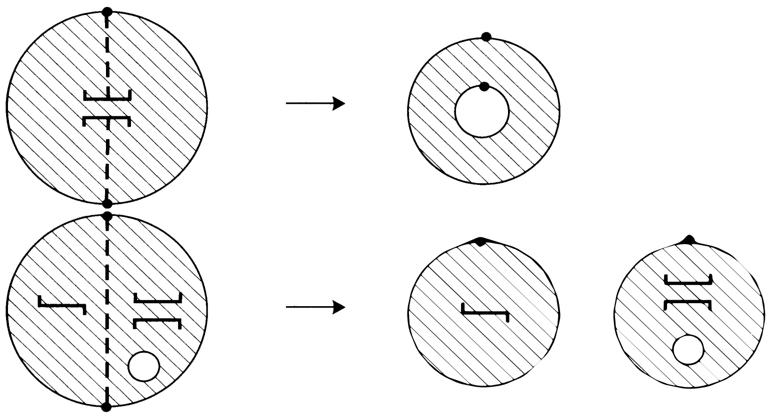

Figure 21

has one more boundary curve than M and may have 1 or 2 connected components; see the two examples in FIGURE 21.

If $\hat{\gamma}$ is orientation reversing, then M' is connected, and has the same number of boundary curves as M; see FIGURE 22. Notice that the condition that $\hat{\gamma}$ is orientation reversing depends only on γ.

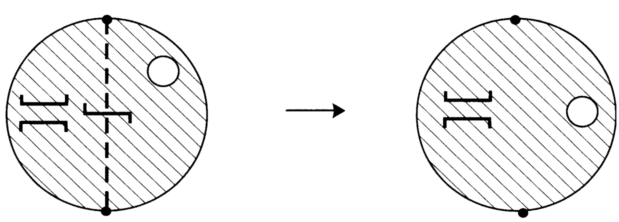

Figure 22

Preparatory results

Consider a game of Cloves with n moves and let M_i denote the surface that results from the ith move. Thus $M_0 = M$ and M_n is the surface at the end of the game. First notice that from the definition of Cloves, one immediately has:

LEMMA 2.

(a) *At each stage of the game, every boundary curve has at least one dot.*

(b) *At the end of the game, each connected component has precisely one boundary curve and it has precisely one dot.*

For Brussels Sprouts, Theorem 1 connects the existence of a winning strategy to the Euler characteristic and the number of crosses. For Cloves, one would want to establish a connection between the existence of a winning strategy and the topological invariants:

- the Euler characteristic $\chi(M)$,
- the number b of boundary curves,
- the number c of connected components,
- the number d of dots.

Unfortunately, one can find games for which the given topological invariants are the same but for which the games have quite different outcomes. For example, the two games in FIGURE 23 both have $b = 2$, $c = 2$, $d = 4$, $\chi = 1$, but the second player is the only possible winner of the game at the top of FIGURE 23, while the first player wins the other game.

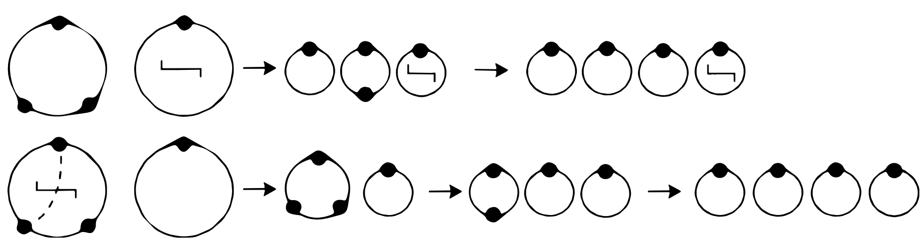

Figure 23

It might seem that in generalizing from Brussels Sprouts to Cloves, we have made the situation harder. Nevertheless, there is a topological description of the game, as we will show. The first important observation is:

LEMMA 3. *If $b = d$, the second player has a winning strategy.*

Proof. The assumption that $b = d$ implies that there is exactly one dot on each boundary curve. The proof is by induction on the integer $j(M) = b - c$. When $j(M) = 0$, each connected component has exactly one boundary curve, and then the hypothesis $b = d$ implies that each connected component has exactly one dot. There is thus no possible move and so the second player wins (without playing a move).

If $j(M) > 0$, the first player does have a possible move, which consists of drawing a curve from a dot on one boundary curve to a dot on another boundary curve lying on the same connected component of M. This is possible since, as $b > c$, there is a connected component with more than one boundary curve. The surface M' that results from this move has $b - 1$ boundary curves, while the number c of connected components is unchanged. Moreover, M' has a (unique) boundary curve with exactly two dots; see FIGURE 24. The second player can now choose a path γ between these two dots that remains very close to the boundary of M'. Cutting along γ gives a disc, with a single dot on its boundary, and a surface M'' homeomorphic to M, with $d - 1$

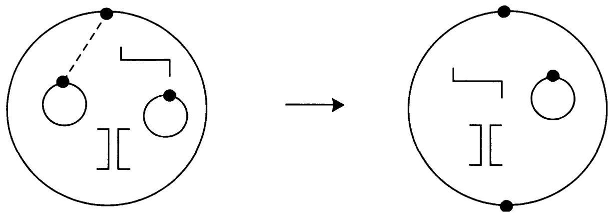

Figure 24

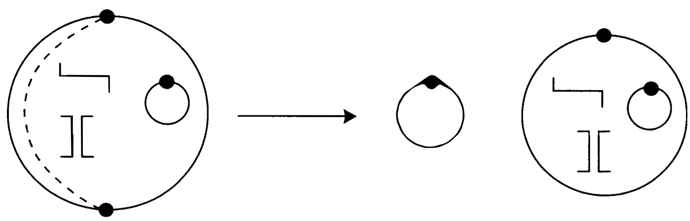

Figure 25

dots; see FIGURE 25. Applying the inductive hypothesis to M'', we obtain the required result. ■

Game theory generalities

At this point, it is useful to pause to consider some general aspects of the game of Cloves. First, Cloves is a *finite game*; that is, given an initial surface M, there is a finite bound on the possible length of the game. In fact, by Lemma 2(b), each game terminates at a surface, M' say, whose connected components have precisely one boundary curve and precisely one dot. So, if M has d dots, $\chi(M') \leq d$. By Lemma 1, if the game lasted for n moves, then $\chi(M') = \chi(M) + n$. Thus $n \leq d - \chi(M)$. This gives an upper bound on the length of a game in terms of the number of dots and the topology of the initial surface M. The following lemma is a key feature of finite games and can be easily established by induction.

LEMMA 4. *For any finite game, either the first player has a winning strategy, or the second player has a winning strategy.*

Another important aspect of Cloves is that it is a game in which *the last player to move is the winner.* Consider the context of all possible abstract two-player games in which the last player to move is the winner. Recall that the *disjunctive sum* of two such games G_1, G_2, is the game $G_1 + G_2$ in which G_1 and G_2 are played side by side, so to speak, and on each player's turn, a "move" is made by making a legal move in just one of the two component games G_1, G_2; see [**6**]. One has the following general fact:

LEMMA 5. *Consider arbitrary finite games G_1, G_2 in which the last player to move is the winner. Assume that the second player has a winning strategy in G_1. Then the first player has a winning strategy in $G_1 + G_2$ if and only if the first player has a winning strategy in G_2.*

Idea of proof. For example, if the second player has a winning strategy in both G_1 and G_2, then the second player can win $G_1 + G_2$ by adopting the following strategy: at each turn, play (sensibly) in whichever game the first player just played in. ■

Remark 3. When the first player has a winning strategy in both G_1 and G_2, one cannot say who will win $G_1 + G_2$ without further information. For some games the first player will have a winning strategy in $G_1 + G_2$, while in other games the second player will have the advantage.

Notice that for two games G_1, G_2 of Cloves, on initial surfaces M_1, M_2 respectively, the game G of Cloves that starts with the disjoint union $M_1 \cup M_2$ is the disjunctive sum $G_1 + G_2$. Thus Lemmas 3 and 5 show that, for a game of Cloves on a surface M, if one of the connected components C of M has the same number of dots as boundary curves, then C plays no role in the game; that is, as far as the outcome of the game is concerned, one may just ignore C. Consequently, for all intents and purposes, we may assume that in every connected component of M, there are more dots than boundary curves. We are now ready to move on to our main result.

Main result

THEOREM 3. *Consider a game of Cloves played on an initial surface M with b boundary curves, d dots and Euler characteristic χ. Assume that in every connected component of M, there are more dots than boundary curves. Then the second player has a winning strategy if and only if b, d, χ are either all odd or all even.*

Proof. The proof is by induction on the integer $j = d - \chi$. Notice that by hypothesis, each connected component has more dots than boundary curves; thus $d \geq b + c$. Hence, as $c \leq b$, we have $d \geq 2c$. By Remark 2 following Theorem 2, $\chi \leq 2c - b$. Thus

$$j = d - \chi \geq d - 2c + b \geq b.$$

Hence the smallest possible value of j is $j = 1$, which occurs when $b = 1$, $c = 1$, $d = 2$, $\chi = 1$; in this case, M is a single disc with two dots on its boundary. The winner is obviously the first player. So Theorem 3 holds for $j = 1$.

Now suppose that $j = d - \chi > 1$ and assume that Theorem 3 holds for all surfaces with smaller j. First suppose that M is orientable. We have:

LEMMA 6. *If M is orientable, the first player wins if and only if $b + d$ is odd (regardless of how the game is played).*

Proof. For an orientable surface M, one has $\chi(M) \equiv b \pmod 2$ by Theorem 2(a). By Lemma 2(b), every connected component of the surface M_n, at the end of the game, has exactly one boundary curve and exactly one dot. Since the number of dots is constant throughout the game, M_n has d connected components and thus $\chi(M_n) \equiv d \pmod 2$. Hence, by Lemma 1,

$$n = \chi(M_n) - \chi(M) \equiv b + d \pmod 2.$$

As the first player wins if and only if n is odd, the proof is complete. ∎

Notice that Lemma 6 proves Theorem 3 in the orientable case. Indeed, one has $\chi \equiv b \pmod 2$ for orientable surfaces, so the condition "b, d, χ all odd or all even" is the same as "b, d both odd or both even," which is the same as "$b + d$ is even."

Now suppose that M is non-orientable. Notice that the condition "b, d, χ are all odd or all even" is equivalent to the condition: $b + d$ and $b + \chi$ are both even. We consider four cases; we show that the first player has a winning strategy in the first three cases, while the second player has a winning strategy in the fourth case.

Case 1. $b + d$ odd, $b + \chi$ even. The first player must first choose a good connected component to play in. He/she should choose a connected component C with $d(C) > b(C) + 1$, if one exists. If there isn't one, then $d(C) = b(C) + 1$ for each connected component C. In this case, since $b(M) + d(M)$ is odd, there is an odd number of connected components. Hence, since $b(M) + \chi(M)$ is even, there is a connected component C for which $b(C) + \chi(C)$ is even. The first player should choose such a connected component.

Having chosen the connected component C, the first player cuts C along a curve joining two dots on the same boundary curve of C, in such a way that the resulting surface C' is the disjoint union of a homeomorphic copy C' of C, with one fewer dot than C, and a disc with a single dot on its boundary; see FIGURE 25. Eliminating this disc, we see that the effect of this move is to replace M by a surface M' for which $b(M') + d(M')$ is even, while $b(M') + \chi(M')$, which has remained unchanged, is still even. Moreover, the integer j has decreased by one.

It remains to apply the inductive hypothesis. To do so, we must ensure that no connected component of our surface has the same number of boundary curves as dots. This is certainly the case for M' if $d(C) > b(C) + 1$ for the connected component C in which the first move was played. In this case, $d(C') > b(C')$ and we can apply the inductive hypothesis to M'. In the other case, from the choice of C, we have $d(C') = b(C')$ and $b(C') + \chi(C') = b(C) + \chi(C)$ is even. By Lemma 3 the second player has a winning stratgey on C', and so by Lemma 5 we can eliminate that component. The elimination of C' doesn't change the parity of $b(M') + d(M')$ or $b(M') + \chi(M')$ and so we can apply the inductive hypothesis to the resulting surface M''. Since the second player has a winning strategy on M' or M'', the first player can adopt that stratgegy to win the game from this point.

Case 2. $b + d$ odd, $b + \chi$ odd. Since $b(M) + \chi(M)$ is odd, M has a connected component C for which $b(C) + \chi(C)$ is odd. In particular, C is non-orientable. The first player's strategy is as follows: choose a boundary curve γ of C with two dots, x and y. Draw a curve from x out to a cross-cap, around the cross-cap and back near x, and hug the boundary of C around to y; see FIGURE 26. Splitting C along γ, we obtain a cross-cap with a single dot on its boundary, and a connected component C' with one fewer dots and one fewer cross-caps than C. Eliminating the cross-cap, the surface M' we obtain has $b(M') + d(M')$ even and $b(M') + \chi(M')$ even. This time the integer j has decreased by 2. If $d(C) > b(C) + 1$, then $d(C') > b(C')$ and we can apply the inductive hypothesis to M'. If $d(C) = b(C) + 1$, then $d(C') = b(C')$. Thus, as $b(C') + \chi(C')$ is even, we can eliminate C' from M' without changing the parity of $b(M') + d(M')$ or $b(M') + \chi(M')$. As in case 1, applying the inductive hypothesis, we conclude that the first player has a winning strategy.

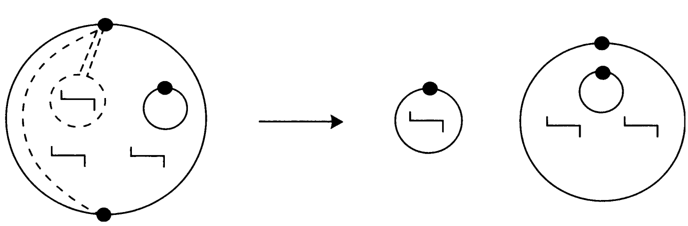

Figure 26

Case 3. $b + d$ even, $b + \chi$ odd. Once again, as $b(M) + \chi(M)$ is odd, M has a non-orientable connected component C for which $b(C) + \chi(C)$ is odd. We choose a boundary curve of C with 2 dots x and y, but this time we join x to y by a curve γ such that splitting along γ gives a connected surface, as in FIGURE 22. For the resulting surface M', j has been reduced by one, b and d are unchanged, and χ has increased by one so $b + \chi$ is now even. As b and d are unchanged, we haven't introduced a connected component with the same number of boundary curves as dots. Hence we may apply the inductive hypothesis, thus giving the first player a winning strategy.

Case 4. $b + d$ even, $b + \chi$ even. We must show that regardless of the move the first player chooses, the second player will find himself in a winning position. There are three possibilities:

Subcase 4a. The first player draws a curve that connects dots on distinct boundary curves, as in FIGURE 24. The resulting surface M' has one fewer boundary curves, and the same number of dots as M. The integer j has decreased by one and $b(M') + d(M')$ is odd. Clearly, we have not created a connected component with the same number of boundary curves as dots; so we can apply the inductive hypothesis (case 1).

Subcase 4b. The first player draws a curve γ that connects dots on the same boundary curve in such a way that splitting along γ produces an extra boundary curve; see the two examples in FIGURE 27.

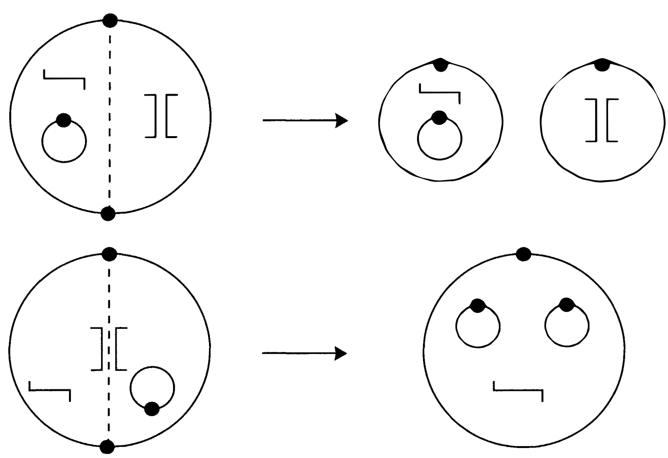

Figure 27

As the number of boundary curves has increased by one, $b(M') + d(M')$ is odd. It is possible that one or both of the newly created connected components has the same number of boundary curves as dots, but eliminating these will not change the parity of $b(M') + d(M')$ nor change the winner of the game, in view of Lemmas 3 and 5. In either case, the integer j has decreased by at least one. So we may apply the inductive hypothesis (case 1 or 2).

Subcase 4c. The first player draws a curve γ that connects dots on the same boundary curve in such a way that splitting along γ doesn't produce an extra boundary curve; see FIGURE 28. In this case, χ decreases by one. Thus $b(M') + d(M')$ is even and $b(M') + \chi(M')$ is odd. As the number of dots and boundary curves is unchanged, we have not created a connected component with the same number of boundary curves as dots, and we can apply the inductive hypothesis (case 3).

This completes the inductive step, and finishes the proof of Theorem 3. ■

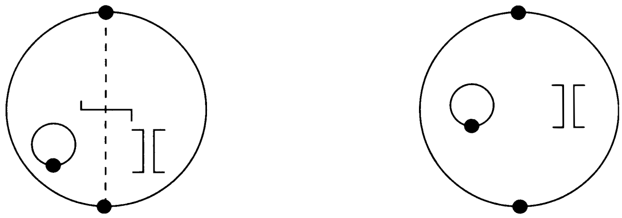

Figure 28

Remark 4. Notice that Theorem 1 is a corollary of Theorem 3. Indeed, for a game of Brussels Sprouts with m crosses on a surface N, the corresponding game of Cloves has $b = m, d = 4m$ and $\chi = \chi(N) - m$. Thus m and $\chi(N)$ are both even if and only if b, d and χ are all even or all odd.

Remark 5. Cloves is closely related to a game called Stars and Stripes [**4**]. In this game, one begins with a general collection of dots with free arms, as in FIGURE 3, and one plays the game as in Brussels Sprouts, except that for each move, when the player draws a new curve γ, the player can choose whether or not to draw a cross on γ. Thus the games of Cloves may be thought of as being between Brussels Sprouts and Stars and Stripes.

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NOTES

Crackpot Angle Bisectors!

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This note is dedicated to the memory of William Thurlow, physician, astronomer, and lifelong student.

A taxi, travelling from the point (x_1, y_1) to the point (x_2, y_2) through a rectangular grid of streets, must cover a distance $|x_1 - x_2| + |y_1 - y_2|$. Using this metric, rather than the Euclidean "as the crow flies" distance, gives an interesting geometry on the plane, often called the *taxicab geometry*. The lines and points of this geometry correspond to those of the normal Euclidean plane. However, the "circle"—the set of all points at a fixed distance from some center—is a square oriented with its edges at 45° to the horizontal (FIGURE 1). As can be seen, there are more patterns of intersection for these than there are in the Euclidean plane.

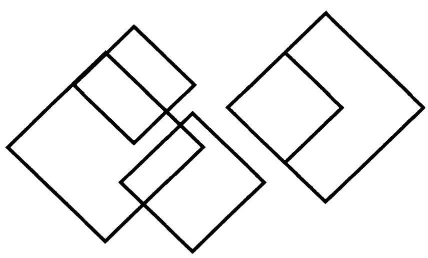

Figure 1 Circles of a taxicab geometry.

There is nothing particularly special about squares and (Euclidean) circles in this context. Any other centrally-symmetric convex body also has a geometry in which it plays the rôle of the circle, with the unit distance in any direction given by the parallel radius of the body. The reader curious about these "Minkowski geometries" is referred to A.C. Thompson's book [**11**].

While taxicab geometry has many applications in advanced mathematics, it is also studied at an elementary level as a foil for Euclidean geometry: a geometry that differs enough from that of Euclid that it enables students to see the function of some fundamental axioms. (As Kipling might have put it, "What do they know of Euclid who only Euclid know?") This use, in undergraduate courses, goes back at least to Martin [**10**] and Krause [**9**], both in 1975. (Byrkit's 1971 article [**5**], while also influential, deals with the taxicab metric on the integer lattice, axiomatically a very different system.)

*Supported by NSERC Canada

In this paper we survey some basic facts about axiomatic taxicab geometry. We give particular consideration to the question of angle measure (an active research area in its own right [**3, 4, 6, 7**]), and show that the taxicab geometry sheds some light on Wantzel's famous result that the angle cannot be trisected by classical methods, a high point of many undergraduate geometry courses.

Following Martin and Krause, we consider the taxicab metric in the context of a set of axioms for Euclidean geometry, based loosely on those of Birkhoff [**1**], which can be summarized as follows. The terms "point" and "line" and the relation "on" are undefined, and the field of real numbers is axiomatized separately. Comments and definitions are interspersed.

INCIDENCE AXIOM. *Two points are on a unique line, and there are three points not all on the same line.*

The unique line through two points A and B is represented as $\overleftrightarrow{AB}$. This axiom allows a line to be identified with the set of points that lie on it.

RULER AXIOM. *For every line l there is a bijection f_l between the points of l and the real numbers.*

A *line segment* is a set of the form $\{x : a \leq f_l(x) \leq b\}$, and its *endpoints* are the points that are mapped to a and b by f_l. A set S is *convex* if it contains any line segment that has both endpoints in S. The *distance* $d(A, B)$ between two distinct points A, B is defined to be $|f(A) - f(B)|$ for the bijection f whose domain is the line $\overleftrightarrow{AB}$. Two line segments $\overline{AB}$ and $\overline{CD}$ are *congruent* if $d(A, B) = d(C, D)$.

SEPARATION AXIOM. *The complement of any line may be partitioned into two convex sets, such that every line segment with one endpoint in each intersects the line.*

A *ray* is any set of points on a line l of the form $\{x : f_l(x) \leq a\}$ or $\{x : f_l(x) \geq a\}$; the point with $f_l(x) = a$ is called the *endpoint*. An *angle* is the configuration consisting of two rays with a common endpoint, not both subsets of a common line. Angles are *supplementary* if they share one ray, and the union of the other two rays is a line.

PROTRACTOR AXIOM. *There exists an additive measure on the angles at each point, such that the measures of two supplementary angles add to π.*

This axiom is actually (see [**10**, §14.2]) provable from the first three. However, it is often included for greater clarity. Angles are defined to be *congruent* if they have the same measure.

SAS CONGRUENCE AXIOM. *If two triangles have two sides and the included angle congruent, then the other side and angles are also congruent.*

PARALLEL AXIOM. *Given a line and a point not on the line, there exists a unique parallel to the line through the point.*

One of the main triumphs of axiomatic geometry is the fact that this axiom set is "categorical": every system obeying it is equivalent to the Euclidean plane. (See, for instance, [**10**, p. 322], for a discussion of this.)

Returning to the taxicab geometry, we see that the first three axioms (and thus the protractor axiom) are valid in the taxicab geometry, as is the parallel axiom. However, while this shows that an angle measure *exists*, it is not unique. In the usual development of an axiomatic system of this type, the function of the SAS axiom is to make both "rulers" and "protractors" invariant under translation (*homogeneous*) and rotation (*isotropic*), thus determining both the metric and the way in which angles are measured. However, no angle measure for the the taxicab geometry can be consistent with the SAS axiom.

This can be proved indirectly, by noting that with any angle measure, the taxicab geometry satisfies **Incidence, Ruler, Separation** and **Parallel**; if it also satisfied **SAS** it would be, as noted above, completely equivalent to the Euclidean geometry. But, whatever angle measure is chosen, two taxicab circles can intersect in a line segment, a thing impossible in the Euclidean plane.

However, there is a more satisfying direct proof. Firstly, we note that the SAS axiom implies (see, for instance, [**10**, §17.3]) the SSS congruence theorem in the presence of the first three axioms. In FIGURE 2, $\triangle OBC$ and $\triangle DOC$ are equilateral and hence (by SSS) have all three angles equal. But $m\angle OCB + m\angle DCO = \pi$, whereas $m\angle BOC + m\angle DOC < \pi$. (A similar, but less general, argument appears on p.195 of [**10**]).

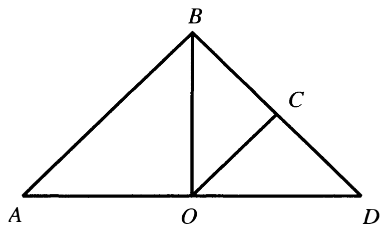

Figure 2 A counterexample to Euc.I.4 in the taxicab geometry

The taxicab geometry, then, demonstrates the function of the **SAS** axiom in somewhat the same way that hyperbolic geometry demonstrates the function of **Parallel**. However, the analogy is not perfect. Denying **Parallel** in the presence of Euclid's other axioms gives a unique alternative, hyperbolic geometry. Denying **SAS** does not let us derive the taxicab geometry. There are many geometries that satisfy the other axioms—for instance, all the two-dimensional Minkowski geometries do so. As observed above, no such geometry can also satisfy **SAS** unless it is Euclidean.

The taxicab metric is not isotropic, but it is homogeneous. Homogeneity can be axiomatized in a "non-**SAS**" geometry in various ways. For instance, **Parallel** could be replaced by the following, based on Euc.I.34:

PARALLELOGRAM AXIOM. *If both pairs of opposite sides of a quadrilateral are parallel, then they and the opposite angles are equal.*

In the presence of **Incidence, Ruler, Separation** and **SAS**, the axioms **Parallel** and **Parallelogram** are equivalent; but without **SAS**, **Parallelogram** is stronger. As well as implying the uniqueness of parallels, it also forces the geometry to be homogeneous; any figure can be translated to any point without distortion of length or angle measure. **Parallelogram** falls short of being a "taxicab **SAS**" axiom, however. It neither implies a taxicab metric (the axiom set {**Incidence, Ruler, Separation, Parallelogram**} is consistent with every Minkowski geometry) nor distinguishes between the various homogeneous taxicab geometries with different protractors.

The fact that obvious axiom sets don't fix the nature of the protractor is presumably one of the reasons why it has been observed (see [**3**, p. 279] or [**6**, p. 32]) that there is no one angle measure that is wholly natural to Minkowski geometry. Martin [**10**, p. 195] and Stahl [**13**, p. 24] consider a geometry in which the taxicab metric is endowed with Euclidean angle measure. Thompson and Dray [**12**] give an alternative model in which angles are measured in "t-radians", units based on the taxicab length subtended on the unit circle. Yet other definitions of angle are also given by A. C. Thompson [**11**] and B. Dekster [**6**]. Moreover, Busemann [**4**] and Glogovškiĭ [**7**] each give different operations generalizing Euclidean angle bisection, which can be taken as bases for angle measurement.

Let's look at some easy theorems of Euclidean geometry, and see how they fare under the new axiom set. The construction of an equilateral triangle (Euc.I.1) is valid in the taxicab geometry (and with a vengeance for any base with a slope of $\pm 45°$, on which there are infinitely many equilateral triangles!) So do the "communication" theorems Euc.I.2 and Euc.I.3. in which a line segment is copied to a specified location and orientation. However, as seen above, the taxicab geometry does not have the SAS congruence property (Euc. I.4). Neither does the *Pons Asinorum* (Euc.I.5, "the base angles of an isosceles triangle are equal") nor its converse hold, the taxi presumably having rendered the donkey obsolete as a means of transportation! Moreover, there is no SSS congruence property; counterexamples are readily found to all of these.

Euclid's next proposition, I.9, is a construction bisecting an angle. Stahl [**13**, p. 59], who follows Martin in using Euclidean angle measure, gives as an exercise "Comment on [Euc. I.9] in the context of [the taxicab geometry]." Given the level of the textbook (undergraduate, with emphasis on prospective teachers), the location of the exercise (in the second chapter), and the lack of comments or hints, the comment is presumably intended to be on the validity of the Euclidean proof (which uses **SAS**) in the taxicab context. However, it is interesting to ask whether some other construction for bisecting "Martin angles" does work.

To pursue this question of angle bisection, we define a "taxi construction" in terms of the following operations:

1. Given two points, construct the straight line through them.
2. Given an ordered pair of points, construct the taxi circle ("diamond") with center at the first and passing through the second.
3. Given two straight lines, construct the point (if any) at which they meet.
4. Given a straight line and a taxi circle, construct their intersection (if any).
5. Given two taxi circles, construct their intersection (if any).

The intersections of a line and a taxi circle, or of two taxi circles, can be one or two points, or (as in FIGURE 1) may consist of a line segment, a point and a line segment, or two line segments. In cases with a line segment, we represent it by its two endpoints (though the fact that it is a segment may be used freely). Construction of any interior points of the intersection must be done separately. Two circles, or a circle and a line, which intersect in two points, will be said to be in *general position*.

We may wonder whether we should also allow the "corners" of a circle, or the horizontal and vertical lines through a point, to be constructed as primitive operations. It turns out that there is no need to do this, as a fairly simple construction using the listed operations gives these.

CONSTRUCTION. *Given a circle, to determine its four corners*

Choose five distinct points A_1, A_2, A_3, A_4, A_5 on the circle. Construct each of the ten lines determined by pairs A_i, A_j, and consider the intersection of each with the circle. By the pigeonhole principle, two of the points must lie on the same side of the circle, and the corresponding intersection will be that entire side. Its endpoints are two adjacent corners of the circle. The lines through each of these corners and the center of the circle intersect the circle again in the other two corners.

As a bonus, we have also constructed the horizontal and vertical lines through the center of the circle! We now come to the main result of the paper.

THEOREM. *There is no construction in the taxicab geometry that will bisect the Euclidean measure of an arbitrary angle.*

Proof. We define a point to be rational if both its coordinates are rational; a line to be rational if it has the form $ax + by = c$ for rational a, b, c; and a taxi circle to be rational if it has rational center and rational radius. It is easily verified that all five elementary constructions, given rational data, return (if anything) one or more rational elements.

Let $O = (0, 0)$, $A = (1, 0)$, and $B = (1, 1)$. The angle $\angle AOB$ has all elements rational; but the ray bisecting it has slope $\tan(\pi/8) = \sqrt{2} - 1$ and is not rational. Thus this angle cannot be bisected in the Martin taxicab geometry. ■

This proof is interesting in its own right; and pedagogically it is very useful as a warmup exercise to prepare students for Wantzel's much more important (but significantly more difficult) proof [**14**] that no Euclidean construction *trisects* an arbitrary angle. (See [**2**], or undergraduate geometry texts such as [**10**] or [**13**], for accessible modern presentations of this result.) In Wantzel's proof, the Hippasian numbers—those that can be obtained using addition, subtraction, multiplication, division, and the square root function—are shown to be closed under Euclidean construction, and to contain values defining a 60° angle but not a 20° angle.

Euclid uses the bisection of an angle in his next proposition, the bisection of an arbitrary straight line. He constructs, on the base $\overline{AB}$, an isosceles triangle $\triangle ABC$, bisects the angle $\angle ACB$, and shows that the bisector also bisects $\overline{AB}$. Clearly, this approach must be abandoned in Martin's taxicab geometry! However, a line segment can be bisected using the taxicab equivalent (FIGURE 3) of an alternative construction ascribed by Proclus to Apollonius [**8**, vol. 1, p. 268].

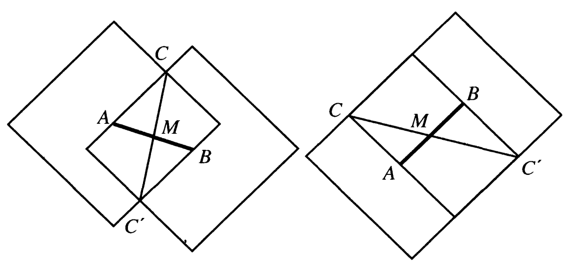

Figure 3 Bisecting the line segment $\overline{AB}$

CONSTRUCTION. *To bisect a line segment using taxicab constructions*

Given a line segment $\overline{AB}$ we construct taxi circles around A through B and around B through A. If the circles are in general position (that is, if the slope of the segment $\overline{AB}$ is not ± 1), we join the two intersection points C, C', and the line $\overline{CC'}$ bisects $\overline{AB}$. Otherwise a slight modification, joining opposite endpoints, effects the same construction.

As a consequence of this construction, we see that in the "t-radian" taxicab geometry of Thompson and Dray, an angle can always be bisected. (Indeed, as a referee of an earlier version of this paper pointed out, it can be trisected, or divided into any number of equal parts.) In many ways, this angle measure is more natural for a taxicab geometry; but the analogy between the failure of angle bisection in Martin's geometry and angle trisection in Euclid's suggests a valuable pedagogical reason for choosing Martin's definition of angle measure, if only one is to be used.

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Quadratic Residues and the Frobenius Coin Problem

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Recently I was struck by the fact that an odd prime p has $(p-1)/2$ quadratic residues mod p and that for relatively prime p and q, there are $(p-1)(q-1)/2$ non-representable Frobenius numbers. I found the presence of $(p-1)/2$ in both expressions curious. Is there some relationship between quadratic residues and the Frobenius numbers that accounts for the presence of $(p-1)/2$ in the two expressions?

As it so happens, there is. Square the non-representable Frobenius numbers for p and q. Mod p, these numbers consist of $q-1$ copies of each of the $(p-1)/2$ quadratic residues mod p, and, mod q, they consist of $p-1$ copies of each of the $(q-1)/2$ quadratic residues mod q. The situation for 5 and 7 is illustrated in the following table. The first row consists of the non-representable Frobenius numbers for 5 and 7, and the second the squares of these numbers. The third and fourth rows are the second row mod 5 and mod 7, respectively.

x	1	2	3	4	6	8	9	11	13	16	18	23
x^2	1	4	9	16	36	64	81	121	169	256	324	529
x^2 mod 5	1	4	4	1	1	4	1	1	4	1	4	4
x^2 mod 7	1	4	2	2	1	1	4	2	1	4	2	4

As we can see, the squares of the non-representable numbers mod 5 consist of six copies each of the two quadratic residues mod 5 (1 and 4) and, mod 7, they consist of four copies each of the three quadratic residues mod 7 (1, 2, and 4). It is not obvious from the table why this might be the case, as there appears to be no pattern to the distribution of the residues.

Before we prove our observation, we should define our terms more carefully. A *quadratic residue* of p is a value of n for which $n \not\equiv 0 \bmod p$ and the equation $x^2 \equiv n \bmod p$ has a solution in x. The quadratic residues mod 5 are 1 and 4 because, mod 5, $1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 4$, and $4^2 \equiv 1$, and any number larger than 5 that is not a multiple of 5 is congruent to one of 1, 2, 3, and 4. One of the most well-known theorems concerning quadratic residues is that an odd prime p has $(p-1)/2$ quadratic residues and $(p-1)/2$ quadratic nonresidues mod p [**1**, p. 179].

Given relatively prime integers p and q, an integer n is *representable* by p and q if there exist nonnonegative integers a and b such that $ap + bq = n$. The *coin problem of Frobenius* is to determine the largest non-representable integer n for a given p and q. The problem is so named because it can be posed like this: A shopkeeper has coins of denominations p and q only. What is the largest amount of money for which the shopkeeper cannot make change? The example given in the table describes the case for five- and seven-cent coins. Using only coins of these two denominations, the shopkeeper can make change for any amount of cents other than those listed in row 1 of the table. The two-coin Frobenius problem—in which coins of two denominations are allowed—was solved by Sylvester [**4**]. His results are that the largest non-representable integer for relatively prime p and q is $(p-1)(q-1)-1$, and there are $(p-1)(q-1)/2$ such non-representable integers. The three-coin Frobenius problem was solved by Selmer and Beyer [**3**]. The Frobenius problem for four or more coin denominations, however, remains unsolved. Guy [**2**, pp. 171–174] contains a discussion of partial results related to the Frobenius coin problem and a long list of references.

It turns out that the set of non-representable Frobenius numbers is a member of a collection of subsets of $\{1, 2, \ldots, pq\}$, all of which produce the quadratic residue phenomena we have observed. We begin our proof of these claims with the following lemma.

LEMMA 1. *If p and q are relatively prime, then any arithmetic sequence of length q with common difference p contains exactly one multiple of q.*

Proof. Let $\{a, a+p, a+2p, \ldots, a+(q-1)p\}$ be an arithmetic sequence of length q with common difference p. Clearly, $\{0, 1, 2, \ldots, q-1\}$ contains exactly one multiple of q. Since p and q are relatively prime, multiplying by p simply permutes this set, mod q. Adding a just permutes the set again, mod q. Thus $\{a, a+p, a+2p, \ldots, a+(q-1)p\}$ contains exactly one multiple of q. ■

With the result of Lemma 1, we can now define a certain class of subsets of $\{1, 2, \ldots, pq\}$ and prove that all of its members produce the observed quadratic residue behavior.

LEMMA 2. *Let p and q be odd primes. Let S be a subset of $\{1, 2, \ldots, pq\}$ with the following properties:*

- *S contains no multiples of p or q.*
- *If x is not a multiple of p or q, then exactly one of x and $pq - x$ is in S.*

Then the squares of the integers in S, mod p, consist of $q-1$ copies of each quadratic residue mod p, and, mod q, they consist of $p-1$ copies of each quadratic residue mod q.

Proof. Let $T = S \cup \{x : x \in \{1, 2, \ldots, pq\}$ and $pq - x \in S\}$. By definition of S, T is $\{1, 2, \ldots, pq\}$ less the multiples of p and q. Also, $\{1, 2, \ldots, pq\}$ can be expressed as $\{a + kp : 1 \leq a \leq p, 0 \leq k \leq q - 1\}$. By Lemma 1, then, the set T consists of $q - 1$ representatives from each of the $p - 1$ nonzero congruence classes of p. Since the squares of a complete residue system mod p produce two copies of each of the quadratic residues mod p [**1**, p. 179], the squares, mod p, of the integers in T consist of $2(q - 1)$ copies of the quadratic residues mod p. As $x^2 \equiv (pq - x)^2$ mod p, the squares of the integers in S, mod p, comprise $q - 1$ copies of each quadratic residue mod p. Swapping the roles of p and q in this argument shows that the squares of S also form $p - 1$ copies of each quadratic residue mod q. ∎

All that remains now is to prove that the non-representable Frobenius numbers have the properties of the set S described in Lemma 2. Since every multiple of p or q is clearly representable, and Sylvester's results [**4**] imply that every integer larger than pq is representable, this reduces to proving the following result. (The result is actually true for relatively prime p and q, not just for p and q prime.)

LEMMA 3. *If p and q are relatively prime, x is an integer such that $0 < x < pq$, and x is not a multiple of p or q, then exactly one of x and $pq - x$ can be represented as a nonnegative combination of p and q.*

Proof. Suppose that $x = ap + bq$ for some nonnegative a and b. Since x is not a multiple of q, we have $0 < a$, and $x < pq$ implies $a < q$. Similarly, $0 < b < p$. Now,

$$pq - x = pq - ap - bq = (q - a)p - bq = -ap + (p - b)q.$$

Both representations of $pq - x$ given here have a negative term. Moreover, any other solution, formed by adding and subtracting kpq from the two terms to obtain

$$pq - x = (q - a - kq)p + (kp - b)q,$$

or

$$pq - x = (kq - a)p + (p - b - kp)q,$$

will necessarily have a negative term for every choice of k. Therefore, $pq - x$ has no nonnegative representation.

Conversely, if x has no nonnegative representation, then, as x is not a multiple of p or q, we must have one negative term and one positive term in any representation. Choose the representation with smallest positive a. Therefore, b must be negative. We have

$$x = ap + bq.$$

If $q \leq a$, we can replace a by $a - q$ and b by $b + p$ to obtain another representation of x. This, however, contradicts the definition of a. Thus $0 < a < q$. Next, if $b \leq -p$, then we have $x < 0$, also a contradiction. Therefore $-p < b < 0$. Consequently,

$$pq - x = pq - ap - bq = (q - a)p - bq,$$

yielding a nonnegative representation of $pq - x$. ∎

Since Lemma 3 shows that the non-representable Frobenius numbers have the properties of the set S described in Lemma 2, we have proved our initial observation:

THEOREM. *Let p and q be odd primes. Then the squares of the non-representable Frobenius numbers for p and q consist, mod p, of $q - 1$ copies of each of the quadratic* *residues mod p, and, mod q, they consist of $p-1$ copies of each of the quadratic residues mod q.*

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Factoring Quartic Polynomials: A Lost Art

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You probably know how to factor the cubic polynomial x^3-4x^2+4x-3 into $(x-3)(x^2-x+1)$. But can you factor the quartic polynomial $x^4-8x^3+22x^2-19x-8$?

Curiously, techniques for factoring quartic polynomials over the rationals are never discussed in modern algebra textbooks. Indeed, Theorem 1 of this note, giving conditions for the reducibility of quartic polynomials, appears in the literature, so far as I know, in only one other place—on page 553 (the very last page) of *Algebra, Part 1* by G. Chrystal [**3**], first published in 1886. Interest in the theory of equations, the subject of this book and many others of similar vintage, seems to have faded, and the factorization theory for quartic polynomials, presented in this note, seems to have been forgotten. Perhaps it is time for a revival!

All polynomials in this note have rational coefficients, that is, all polynomials are in $\mathbb{Q}[x]$. Moreover, we are interested only in factorizations into polynomials in $\mathbb{Q}[x]$. The factorization $x^2-2=(x+\sqrt{2})(x-\sqrt{2})$ is not of this type since $x+\sqrt{2}$ and $x-\sqrt{2}$ are not in $\mathbb{Q}[x]$. In our context, x^2-2 has no nontrivial factorizations and so is *irreducible*. A polynomial, such as $x^3-4x^2+4x-3=(x-3)(x^2-x+1)$, which has a nontrivial factorization is said to be *reducible*. For a nice general discussion about the factorization of polynomials over $\mathbb{Q}$, see [**1**].

Basic tools for factoring polynomials are the following:

- *Factor Theorem:* Let $f\in\mathbb{Q}[x]$ and $c\in\mathbb{Q}$. Then c is a root of f (that is, $f(c)=0$) if and only if $x-c$ is a factor of $f(x)$.
- *Rational Roots Theorem:* Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with integer coefficients $a_n, a_{n-1},\ldots,a_0$. If p/q is a rational number in lowest terms such that $f(p/q)=0$, then p divides a_0 and q divides a_n.

These theorems suffice to factor any quadratic or cubic polynomial since such a polynomial is reducible if and only if it has a root in $\mathbb{Q}$. Finding such a root is made easy by the rational roots theorem, and then long division yields the corresponding factorization.

On the other hand, a quartic polynomial may factor into a product of two quadratic polynomials but have no roots in $\mathbb{Q}$. For example, $f(x) = (x^2 - 2)(x^2 - 2)$ has no roots in $\mathbb{Q}$ but obviously factors. Thus to determine whether or not a quartic polynomial without rational roots is reducible, we need to know whether it factors into a product of two quadratic polynomials. Theorem 1 shows that this question can be answered using an associated cubic polynomial called the resolvent.

To simplify our presentation we will consider only polynomials in reduced form: If $f(x) = ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Q}[x]$ (with $a \neq 0$) is an arbitrary quartic polynomial, then the *reduced form* of f is the polynomial $f(x - b/4a)/a$. For example, the reduced form of $f(x) = x^4 - 8x^3 + 22x^2 - 19x - 8$ is $f(x+2) = x^4 - 2x^2 + 5x - 6$. The reduced form has leading coefficient one and no degree three term. It is easy to see how a factorization of the reduced form gives a factorization of the original polynomial (see Example 4). Thus we lose no generality in the following theorem by assuming that f is already in the reduced form $f(x) = x^4 + cx^2 + dx + e$. In this circumstance, the *resolvent* of f is the cubic polynomial

$$R(z) = z^3 + 2c\,z^2 + (c^2 - 4e)\,z - d^2.$$

Since it is easy to calculate the roots of f once it has been factored, it is no surprise that the resolvent also appears in the many published methods for finding the roots of quartic polynomials (see, for example, [**2**]).

In what follows we write $\mathbb{Q}^2 = \{s^2 \mid s \in \mathbb{Q}\}$ for the set of squares in $\mathbb{Q}$.

THEOREM 1. *The quartic polynomial $f(x) = x^4 + cx^2 + dx + e \in \mathbb{Q}[x]$ factors into quadratic polynomials in $\mathbb{Q}[x]$ if and only if (at least) one of the following holds:*

(A) The resolvent R has a nonzero root in $\mathbb{Q}^2$.

(B) $d = 0$ and $c^2 - 4e \in \mathbb{Q}^2$.

Proof. Suppose f factors as

$$f(x) = (x^2 + hx + k)(x^2 + h'x + k'), \tag{1}$$

with $h, h', k, k' \in \mathbb{Q}$. Multiplying (1) out and matching coefficients we get

$$0 = h + h', \qquad e = kk', \tag{2}$$

$$d = hk' + h'k, \qquad c = hh' + k + k'. \tag{3}$$

In particular, $h' = -h$. The equations in (3) are linear in k and k' and can be solved to yield

$$2hk = h^3 + ch - d, \quad 2hk' = h^3 + ch + d. \tag{4}$$

From $e = kk'$ and (4) we get

$$4h^2e = (2hk)(2hk') = (h^3 + ch - d)(h^3 + ch + d). \tag{5}$$

Multiplying this out we get

$$h^6 + 2c\,h^4 + (c^2 - 4e)\,h^2 - d^2 = 0, \tag{6}$$

and so h^2 is a root of the resolvent R. If $h \neq 0$, then (A) of the theorem holds. Otherwise, $h = 0$ and (6) implies that $d = 0$ and, from (2) and (3), we get $c^2 - 4e = (k + k')^2 - 4kk' = (k - k')^2 \in \mathbb{Q}^2$. Thus, in this case, (B) of the theorem holds.

Now suppose that the resolvent R has a nonzero root in $\mathbb{Q}^2$. Then there is some nonzero $h \in \mathbb{Q}$ such that (6) holds. Set

$$h' = -h, \quad k = \frac{1}{2h}(h^3 + ch - d), \quad k' = \frac{1}{2h}(h^3 + ch + d). \tag{7}$$

Then $h', k, k' \in \mathbb{Q}$ and, since (5) follows from (6), the equations (2) and (3) hold. Thus f factors into quadratic polynomials in $\mathbb{Q}[x]$ as in (1).

Suppose that $d = 0$ and $c^2 - 4e \in \mathbb{Q}^2$. Then $c^2 - 4e = s^2$ for some $s \in \mathbb{Q}$. Set

$$h = h' = 0, \quad k = (c + s)/2 \text{ and } k' = (c - s)/2. \tag{8}$$

Then $h, h', k, k' \in \mathbb{Q}$ and $k + k' = c$, $kk' = (c^2 - s^2)/4 = e$, $f(x) = (x^2 + k)(x^2 + k')$, and so once again f factors into quadratic polynomials in $\mathbb{Q}[x]$. ∎

From the proof of this theorem we can extract an algorithm for factoring a quartic polynomial f in reduced form. First, using the rational roots theorem, look for a rational root of f. If $c \in \mathbb{Q}$ is such a root, then, by the factor theorem, we know that $f(x) = (x - c)\, g(x)$ for some cubic polynomial g (which can be determined by long division). If f has no rational roots, we look for rational roots of the resolvent R. If $h^2 \in \mathbb{Q}^2$ is a nonzero root of R, then condition (A) of Theorem 1 holds, and (7) and (1) give a factorization of f. If condition (B) of Theorem 1 holds, then equations (8) and (1) determine a factorization of f. If these steps fail to produce a factorization, then f is irreducible.

EXAMPLE 1. Let $f(x) = x^4 + x^2 + x + 1$. Then neither f nor the resolvent $R(z) = z^3 + 2z^2 - 3z - 1$ has a rational root. Thus f is irreducible.

EXAMPLE 2. Let $f(x) = x^4 + 2x^2 + 5x + 11$. Then f has no rational roots, and the resolvent $R(z) = z^3 + 4z^2 - 40z - 25$ has one rational root, namely 5, which is not in $\mathbb{Q}^2$. Thus f is irreducible.

EXAMPLE 3. Let $f(x) = x^4 - 12x^2 - 3x + 2$. Then f has no rational roots, and the resolvent $R(z) = z^3 - 24z^2 + 136z - 9$ has one rational root, namely $9 \in \mathbb{Q}^2$. Thus f is reducible. Setting $h = \sqrt{9} = 3$ in (7) and (1) we get $f(x) = (x^2 + 3x - 1)(x^2 - 3x - 2)$.

EXAMPLE 4. Let $f(x) = x^4 - 8x^3 + 22x^2 - 19x - 8$, the motivating example from the beginning of this note. Then f has no rational roots. The reduced form of this polynomial is $f(x + 2) = x^4 - 2x^2 + 5x - 6$, and its resolvent is $R(z) = z^3 - 4z^2 + 28z - 25$ with one rational root, namely, $1 \in \mathbb{Q}^2$. Thus f is reducible. Setting $h = \sqrt{1} = 1$ in (7) and (1) we get $f(x + 2) = (x^2 + x - 3)(x^2 - x + 2)$ and so $f(x) = (x^2 - 3x - 1)(x^2 - 5x + 8)$.

We conclude by investigating the interesting special case when $f(x) = x^4 + cx^2 + e$. If $r \in \mathbb{Q}$ is a root of $f(x) = x^4 + cx^2 + e$ then so is $-r$, and $x^2 - r^2 \in \mathbb{Q}[x]$ divides f. Thus f is reducible if and only if it factors into two quadratic polynomials. Since $d = 0$, the resolvent of f is

$$R(z) = z\left(z^2 + 2c\,z + (c^2 - 4e)\right),$$

with roots $0, -c \pm 2\sqrt{e}$. Theorem 1 now provides a test for the irreducibility of f:

THEOREM 2. [**4**, Theorem 2] *A quartic polynomial $f(x) = x^4 + cx^2 + e \in \mathbb{Q}[x]$ is reducible if and only if $c^2 - 4e \in \mathbb{Q}^2$ or $-c + 2\sqrt{e} \in \mathbb{Q}^2$ or $-c - 2\sqrt{e} \in \mathbb{Q}^2$. For the conditions involving $\sqrt{e}$ to hold it is, of course, necessary that $e \in \mathbb{Q}^2$.*

EXAMPLE 5. If $f(x) = x^4 - 3x^2 + 1$, then $c = -3$ and $e = 1$. We have $c^2 - 4e = 5 \notin \mathbb{Q}^2$, $-c + 2\sqrt{e} = 5 \notin \mathbb{Q}^2$ and $-c - 2\sqrt{e} = 1 \in \mathbb{Q}^2$. Thus f is reducible. To calculate the factorization we set $h = 1$ in (7) and (1) to get $f(x) = (x^2 + x - 1)(x^2 - x - 1)$.

EXAMPLE 6. If $f(x) = x^4 - 16x^2 + 4$, then $c = -16$ and $e = 4$. We have $c^2 - 4e = 240 \notin \mathbb{Q}^2$, $-c + 2\sqrt{e} = 20 \notin \mathbb{Q}^2$ and $-c - 2\sqrt{e} = 12 \notin \mathbb{Q}^2$, and so f is irreducible.

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Butterflies in Quadrilaterals: A Comment on a Note by Sidney Kung

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In the October 2005 issue of *Mathematics Magazine*, Sidney Kung published a note on a theorem on butterflies inscribed in a quadrilateral which bears remarkable similarity to the usual Butterfly Theorem (see [**5**]). In this note, we will show that this similarity is not a coincidence. In fact, Kung's Theorem really is the usual Butterfly Theorem in disguise. To see this, it is easiest to try to prove Kung's Theorem using projective geometry. Indeed, if we insist on using the standard toolbox of projective geometry, it might not even be possible to prove the theorem without reducing it to some version of the usual Butterfly Theorem no matter what the kind of tools we allow—at least we have not been able to find such a proof. For the purposes of this note we will only use a few basic tools of the field, specifically the notion of an involution on a (projective) straight line and Desargues' Involution Theorem.[1]

We start with a reformulation of the Butterfly Theorem in terms of projective geometry. Consider a self-intersecting quadrilateral $AB'A'B$ (the "butterfly" in FIGURE 1) inscribed in a conic section $\mathcal{C}$. Let I be the point of intersection of the sides $A'B$ and AB'. Now draw an arbitrary line through I and let C, C' be the points of intersection of the line with the conic section. By Desargues' Involution Theorem, the family of conic sections circumscribing $AB'A'B$ defines an involution on the line CC', with I a fixed point of this involution and C, C' a conjugate pair (i.e. they are each other's images under the involution).[2] This completely determines the involution: Since the fixed points of an involution are in harmonic position with respect to any conjugate

[1]Most of the theory needed can be found in any textbook on projective geometry. A classic is [**2**]. For a discussion of Desargues' Theorem, see also Problem 63 of [**3**, pp. 265–273] . Within the canon of projective geometry, Desargues' Theorem is usually derived from Steiner's Theorems and some other fundamental tools. With a little bit of analytic geometry, however, the theorem is almost immediately proved. See also the next footnote.

[2]Intuitively, this follows from the fact that five points determine a conic section. Therefore, for any point on CC', there is a unique conic passing through the point and circumscribing $AB'A'B$. This conic intersects CC' in only one other point (possibly the same point). Thus all the points on CC' come in pairs.

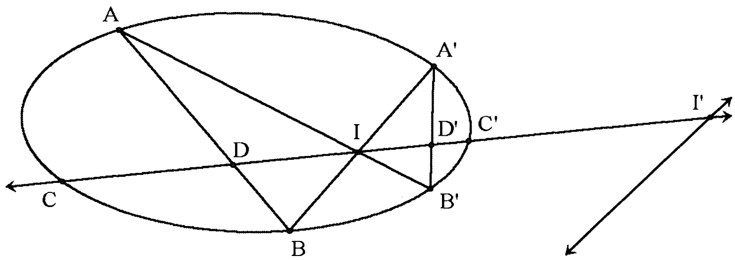

Figure 1 A Butterfly in An Ellipse

pair of points, we can determine the second fixed point I'. In fact, I' is the point of intersection of CC' with the polar line of I with respect to the conic section $\mathcal{C}$. The involution on CC' is the circular inversion with respect to the circle that has II' for a diameter. It is easily verified that for this inversion one has for all conjugate points P, P' the equality

$$\frac{1}{IP} + \frac{1}{IP'} = \frac{2}{II'},$$

where IP etc. are to be taken as "directed" distances (i.e. $IP = -PI$). Since C and C' form a conjugate pair while $D = AB \cap CC'$, $D' = A'B' \cap CC'$ form another one, it follows that

$$\frac{1}{IC} + \frac{1}{IC'} = \frac{1}{ID} + \frac{1}{ID'}$$

or, in the case of our drawing,

$$\frac{1}{|IC|} - \frac{1}{|IC'|} = \frac{1}{|ID|} - \frac{1}{|ID'|},$$

where $|IC|$ denotes the absolute distance between I and C and so on. This is a slight generalization of the version of the Butterfly Theorem that Kung refers to in his note (see [**5**, p. 314]). The usual Butterfly Theorem follows by assuming that I is the midpoint of CC'. A similar line of argument can be used to prove Murray Klamkin's generalization of the Butterfly Theorem along with most of the other variations on the usual Butterfly Theorem that are discussed on Alexander Bogomolny's Cut-The-Knot website (see [**1**], [**4**]).

The connection with Kung's Theorem is illustrated in FIGURE 2. For a (convex) quadrilateral $ABCD$, let I be the point of intersection of the diagonals AC and BD. For any line passing through I, let G and G' the points of intersection with DA and CB, respectively. Now consider the family of conic sections passing through D, B, G and G'. Since five points determine a conic section, there is a unique conic section $\mathcal{C}_H$ in this family that passes through a given point H on AB. Next draw the line HI. Let H' be the point of intersection of HI with DC and let $\bar{H}$ be the second point of intersection of HI with $\mathcal{C}_H$. The point $\bar{H}$ will be the image of the point H under the involution on HI defined by the family of conic sections through D, B, G and G'. But this involution has I for a fixed point and the points $H'' = HI \cap CB$ and $H''' = HI \cap AD$ for a conjugate pair. This uniquely determines the involution. However, under the involution induced on HI by the family of conic sections circumscribing $ABCD$ the same points H'' and H''' form a conjugate pair as well, while I again is a fixed point. In other words, the two families of conic sections give rise to the same

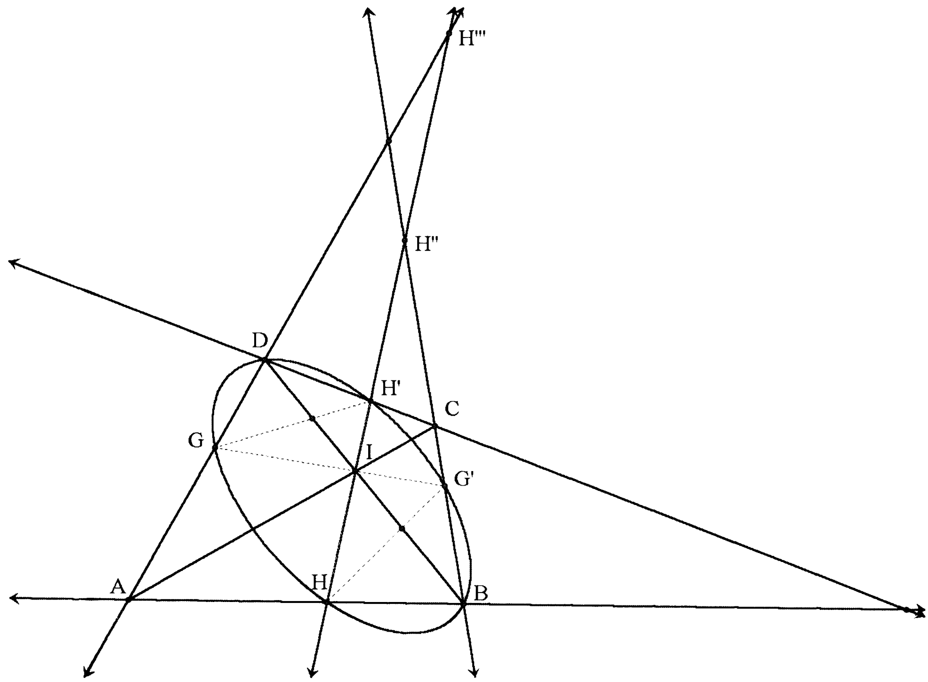

Figure 2 A Butterfly in A Quadrilateral

involution on HI. According to this second definition of the involution, the image of H is H'. It follows that H' and $\bar{H}$ coincide. Consequently D, B, G, G', H and H' all lie on the same conic. Alternatively, if we wanted to use a slightly more heavy-duty tool of projective geometry, we could have noted that $GD \cap HB = A$, $DH' \cap BG' = C$, and $H'H \cap G'G = I$ are collinear. Therefore, by the converse of Pascal's Theorem, it follows that there is a conic that circumscribes the hexagon $GDH'HBG'$.[3] Either way, this means that we have the set-up of the Butterfly Theorem as outlined at the beginning of this note. Kung's Theorem immediately follows.

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5. Sidney Kung, A Butterfly Theorem for Quadrilaterals, this MAGAZINE, **78** (2005) 314–316.

[3]The author would like to thank one of the referees for pointing this out. This observation actually suggests an elementary way to prove the converse of Pascal's Theorem. On Pascal's Theorem, see Problem 61 of [**3**, pp. 257–261].

The Dottie Number

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The Dottie number was the nickname among my graduate school friends for the unique real root of $\cos(x) = x$. The story goes that Dottie, a professor of French, noticed that whenever she put a number in the calculator and hit the cos button over and over again, the number on the screen always went to the same value, about 0.739085.... She asked her math-professor husband why the calculator did this no matter what number she started with. He looked. He tried it. He said he had no idea, at least not that day. The next day he realized not only what was happening, but that his wife had found a beautiful, simple example of a global attractor.

Dottie was computing the sequence defined by the recursion relation, $s_{n+1} = \cos(s_n)$. This sequence has a unique fixed point at the root of $\cos(x) = x$ where x is, of course, expressed in radians. Moreover, the domain of attraction for this fixed point was the entire real line. So any value used for s_0 will generate a sequence that converges to the same root.

In my own teaching, I have been inserting the Dottie number story into my courses. I use the story to teach students that when they get stuck on a problem, it is okay to stop and come back to the problem later, refreshed. I also follow up the story with a homework problem related to the Dottie number. For convenience, I will denote the Dottie number by d in this paper.

I have first semester Calculus students demonstrate that the Dottie number is the unique root of $\cos(x) = x$. They then apply the Intermediate Value Theorem to the function, $f(x) = x - \cos(x)$, on the interval $(-\pi/2, 3\pi/2)$ to show the existence of d. Using Rolle's Theorem, they give a proof by contradiction that d is the unique root on that interval. Then they have to show that there are no roots outside the interval.

In differential equations, I show students how to use Euler's method to find roots of a function. The differential equation $x' = f(x)$ has fixed points or equilibrium solutions at the roots of $f(x)$. Using Euler's method, a numerical approximation to $x' = f(x, t)$ for the initial condition $x(t_0) = x_0$ can be found from the series, $t_{n+1} = t_n + \Delta t$ and $x_{n+1} = x_n + \Delta t f(x_n, t_n)$ where Δt is a given parameter. I have students find a numerical approximation to the Dottie number by finding a solution to $x' = \cos(x) - x$ with $x(0) = 0$ and $\Delta x = 1$. I ask them to write a report about what they are doing and what they found.

In my Problems in Math course or an equivalent Advanced Calculus course, I talk about inverse power series. Looking for a good problem, I discovered that the Dottie number has a power series in odd powers of π. I got my students to prove that the Dottie number can be written in the form

$$\mathsf{d} = \sum_{n=0}^{\infty} a_n \pi^{2n+1}$$

where each coefficient, a_n is rational. For the function $f(x) = x - \cos(x)$, $f(\mathsf{d}) = 0$. Hence $f^{-1}(0) = \mathsf{d}$ where f^{-1} is defined in the interval $(-\pi/2, 3\pi/2)$. Using the Taylor series for f^{-1} about $\pi/2$, I have students then find the first two coefficients,

a_0 and a_1 in the power series for $f^{-1}(0)$, the Dottie Number. This method requires students to compute the nth derivative of f^{-1} at $\pi/2$ in terms of the first n derivatives of f at $\pi/2$. By the way, $a_0 = 1/4$ and $a_1 = -1/768$.

In my Chaos course I make sure to tell the story of the Dottie number right away—without the punch line—and ask them what is going on. They repeat the experiment themselves and make conjectures. We come back to this example every time we learn one new element of finding attracting fixed points and domains of attraction.

In my complex variables class, we show that $\cos(z) = z$ has infinitely many complex roots that come in complex conjugate pairs (except for the Dottie number). We do this by studying the complex form of $\cos(z) = (e^{iz} + e^{-iz})/2$. Later in the complex variables course, I introduce complex dynamics. When we get to Julia sets, we compute the Julia set numerically for $\cos(z)$ and see the domain of attraction for d. The other roots are repelling.

It is unlikely that the Dottie number will enter the annals of great constants alongside e, π, the Golden Ratio and many others. However, the Dottie number and its story might make good teaching elements for others out there. I also imagine there are many other interesting facets of the Dottie number yet to be discovered. I look forward to hearing about what you find.

d = 0.73908 51332 15160 64165 53120 87673 87340 40134 11758 90075
74649 65680 63577 32846 54883 54759 45993 76106 93176 65318...

Proof Without Words: Alternating Sums of Squares of Odd Numbers

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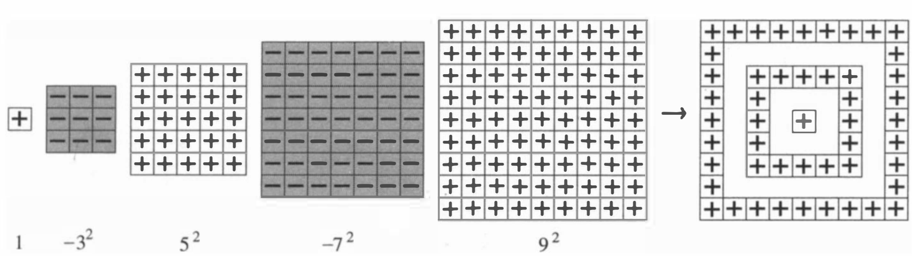
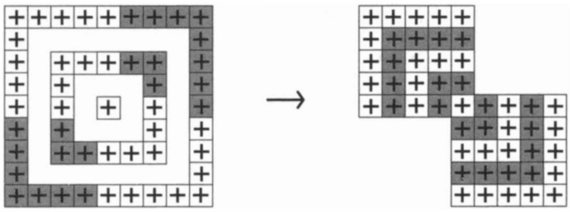
If n odd:

$$\sum_{k=1}^{n} (2k-1)^2 (-1)^{k-1} = 2n^2 - 1$$

E.g. $n = 5$:

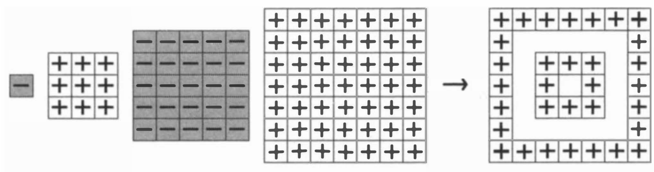


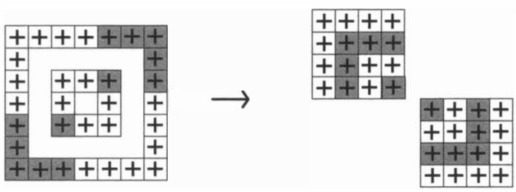


If n even:

$$\sum_{k=1}^{n} (2k-1)^2 (-1)^k = 2n^2$$

E.g. $n = 4$:





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An Integral Domain Lacking Unique Factorization into Irreducibles

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Long ago I studied at Adelphi University. Teaching there at that time was Donald Solitar. An associate told me that Solitar was working on an abstract algebra textbook whose selling point would be lots of great examples. When I asked for one, I was shown the following, which, until recently, I had never seen in print. I have now learned that it has appeared in a book by Rotman [1]. Whatever its provenance, this example deserves a wider audience.

Consider the set, S of polynomials in one variable over the integers with zero coefficient on the linear term. That is to say consider:

$$S = \{a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \cdots + a_2 \cdot x^2 + a_0\}$$

Now it's easy to verify that S is an integral domain. But the delightful surprise is that this integral domain does *not* have unique factorization into *irreducibles* (nonunit elements x such that if $x = yz$ then y or z is a unit) and that this is clear immediately from what follows. Consider x^6. This can be written as $x^2 \cdot x^2 \cdot x^2$ or $x^3 \cdot x^3$. Both x^2 and x^3 are clearly irreducible in S. And since these two factorizations into irreducibles contain different numbers of factors, they are distinct.

As a bonus we also find that x^2 and x^3 are not *prime* illustrating that *prime* and *irreducible* are separate concepts. For example, x^2 divides $x^3 \cdot x^3$ while not dividing either factor.

REFERENCE

1. Joseph J. Rotman, *Advanced Modern Algebra*, Pearson, 2002, p. 330.

Proof Without Words: Alternating Sum of an Even Number of Triangular Numbers

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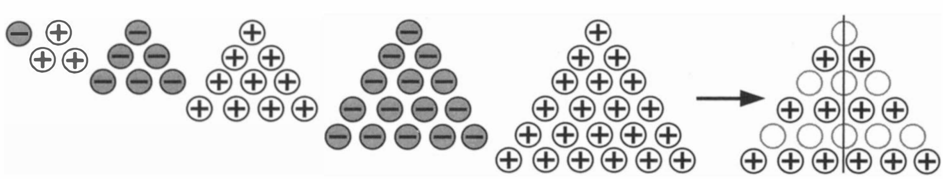
$$t_k = 1 + 2 + \ldots + k \Rightarrow \sum_{k=1}^{2n} (-1)^k t_k = 2t_n$$

E.g., $n = 3$:



$-t_1 + t_2 \quad -t_3 \quad +t_4 \quad -t_5 \quad +t_6 \quad = \quad 2t_3$

NOTE. For a "proof without words" of a similar statement—alternating sums of an odd number of triangular numbers—see Roger B. Nelsen, this MAGAZINE, Vol. 64, no. 4 (1995), p. 284.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by July 1, 2007.

1761. *Proposed by Steve Butler, University of California San Diego, La Jolla, CA.*

For integer $n \geq 2$ define the sets

$$A(n) = \{(k, l) : 1 \leq k \leq l \leq n,\ k + l \leq n,\ \text{and } \gcd(k, l) = 1\}$$
$$B(n) = \{(k, l) : 1 \leq k \leq l \leq n,\ k + l > n,\ \text{and } \gcd(k, l) = 1\},$$

where $\gcd(k, l)$ denotes the greatest common divisor of the integers k and l. Prove that $A(n)$ and $B(n)$ have the same cardinality.

1762. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, New York, NY.*

Let n be an integer with $n \geq 2$. Prove that for any even integer k, there exist odd primes p and q such that $p + q \equiv k \pmod{n}$.

1763. *Proposed by Joshua T. Wood and William P. Wardlaw, U. S. Naval Academy, Annapolis, MD.*

Let ℓ_1 and ℓ_2 be two lines in three space, let the distance between ℓ_1 and ℓ_2, measured along a mutual perpendicular to both lines, be d, and let θ be the angle determined by the direction vectors of ℓ_1 and ℓ_2. A line segment of length a lies on ℓ_1 and a line segment of length b lies on ℓ_2. Determine the volume of the convex hull of these two segments.

1764. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI.*

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Elgin Johnston, Problems Editor, Department of Mathematics, Iowa State University, Ames IA 50011, or mailed electronically (ideally as a LaTeX file) to ehjohnst@iastate.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

For positive integer n, let $g_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$. Prove that

$$\lim_{n\to\infty}\left(\frac{g_n^{\gamma}}{\gamma^{g_n}}\right)^{2n} = \frac{e}{\gamma},$$

where γ is the Euler-Mascheroni constant.

1765. *Proposed by Eugene A. Herman, Grinnell College, Grinnell, IA.*

An object in 3-space is translated by a fixed vector **t** and then rotated using a rotation matrix whose axis of rotation has unit direction vector **a** and for which the angle of rotation in a plane perpendicular to **a** is $\theta = \frac{\pi}{n}$, where n is a positive integer. This translation-rotation move is repeated for a total of $2n$ times. When this is done, what are the position and orientation of the object relative to its initial position and orientation?

Quickies

Answers to the Quickies are on page 82.

Q967. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let f be a derivative on $I = [0, 1]$. Prove that for each $\epsilon > 0$, there is a function $\delta : I \to (0, \infty)$ such that if $x, y \in I$ with $|x - y| < \delta(x)$ and $|x - y| < \delta(y)$, then $|f(x) - f(y)| < \epsilon$. (To say that f is a derivative on I means that there is a real valued function F defined on I such that $F'(x) = f(x)$ for all $x \in I$.)

Q968. *Proposed by Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY.*

Let a, b, and c be positive real numbers, all greater than 1. Prove that

$$\log_a abc + \log_b abc + \log_c abc \geq 9,$$

and determine when equality occurs.

Solutions

Rational vertices for a triangle **February 2006**

1736. *Proposed by Claude Bégin, Mont-St-Hilaire, Québec, Canada.*

Is there a triangle all of whose vertices are points with rational coordinates on the circle $x^2 + y^2 = 1$ and whose vertex angles are 45°, 60°, and 75°?

Solution by Roy Barbara, American University of Beirut, Beirut, Lebanon.

We prove the following more general result:

Let T be a triangle in the plane. If at least one angle of T has an irrational tangent, then the vertices of the triangle cannot all have rational coordinates.

Assume that such a triangle T does exist, and that it has vertex angles α, β, γ, and that $\tan\alpha$ irrational. Then at least one of $\tan\beta$ and $\tan\gamma$ must also be irrational because if not, then

$$\tan\alpha = \tan(\pi - \beta - \gamma) = -\frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma}$$

is rational (with the obvious interpretations if $\tan\beta$ or $\tan\gamma$ is infinite.) Thus we may assume that $\tan\beta$ is also irrational (and finite) and that $\alpha \le \beta$. It follows that $0 < \alpha < \frac{\pi}{2}$. Now translate T so the vertex with angle α is at the origin O. The translated triangle also has vertices with rational coordinates. Let the other two vertices be $P = (a, b)$ and $Q = (c, d)$, and let X be a point on the positive x-axis. If $\tan(\angle XOP) = \pm\infty$, then $\tan(\angle XOQ)$ is finite because $\angle POQ$ has magnitude of less than $\frac{\pi}{2}$. Hence

$$\tan\alpha = \pm\tan(\angle XOP - \angle XOQ) = \pm\cot(\angle XOQ) = \pm\frac{c}{d},$$

is rational, contradicting our assumptions about α. If $\tan(\angle XOQ) = \pm\infty$, then we arrive at a similar contradiction. Finally, if $\tan(\angle XOP)$ and $\tan(\angle XOQ)$ are both finite, then

$$\tan\alpha = \pm\tan(\angle XOP - \angle XOQ) = \pm\frac{\tan(\angle XOP) - \angle XOQ)}{1 + \tan(\angle XOP)\tan(\angle XOQ)} = \frac{\frac{b}{a} - \frac{d}{c}}{1 - \frac{b}{a}\frac{d}{c}}$$

is again rational. This contradicts the assumption that $\tan\alpha$ is irrational, and completes the proof of the result.

Also solved by, "ABC" Student Problem Solving Group, Herb Bailey, Michel Bataille (France), Brian D. Beasley, Jany C. Binz (Switzerland), Jean Bogaert (Belgium), Grady Bullington, Bruce S. Burdick, Robert Calcaterra, Cal Poly Problem Solving Group, Minh Can, Doug Cashing, Alper Cay and Uzman Dersane (Turkey), John Christopher, Con Amore Problem Group (Denmark), Calvin A. Curtindolph, Knut Dale (Norway), Prithwijit De (Ireland), Jim Delany, Robert L. Doucette, Eric Duchon and David Lovit, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Dmitry Fleischman, Leon Gerber, Leon Harkleroad, James R. Henderson, George W. Hukle, Elias Lampakis (Greece), Tom Leong, Aaron Lieberman, S. C. Locke, Luk, Sai Luk (Hong Kong), Paul Martin, Nicholas Mecholsky, Shoeleh Mutameni, José H. Nieto (Venezuela), Bill Post, Raúl Simón (Chile), Seshadri Sivakumar (Canada), Tony Tam, H. T. Tang, Marian Tetiva (Romania), Loretta FitzGerald Tokoly, Thomas Vanden Eynden, Michael Vowe (Switzerland), Paul Weisenhorn (Germany), Doug Wilcock, Jerry Seaton and Bill Yankosky, John B. Zacharias (Australia), and the proposer.

A perfect hypotenuse **February 2006**

1737. *Proposed by Michael Z. Spivey, The University of Puget Sound, Tacoma, WA.*

A Pythagorean triple is an ordered triple (a, b, c) of positive integers satisfying $a^2 + b^2 = c^2$. The number c is called the hypotenuse of the Pythagorean triple.

(a) Prove that an even perfect number cannot be the hypotenuse of a Pythagorean triple.

(b) Prove that if there is an odd perfect number, then it is the hypotenuse of a Pythagorean triple.

Solution by Robert L. Doucette, McNeese State University, Lake Charles, LA.

(a) An even perfect number n must be of the form $2^{k-1}(2^k - 1)$ with $2^k - 1$ prime. In addition, a number is the hypotenuse of a Pythagorean triple if and only if it is a multiple of a number of the form $t = r^2 + s^2$ where r and s are relatively prime positive integers and have opposite parity. Thus, $t \equiv 1 \pmod 4$. Because $2^k - 1 \equiv 3 \pmod 4$, it follows that the number n cannot be a multiple of such a number t.

(b) By a result of Euler, an odd perfect number n must be of the form $n = p^\alpha m^2$ with p prime and $p \equiv \alpha \equiv 1 \pmod 4$. By a result of Fermat, n can be represented as the sum of two squares if and only if every prime factor congruent to 3 modulo 4 in the prime factorization of n occurs with an even exponent. It follows that if n is an odd perfect number, then $n = x^2 + y^2$ for some positive integers x and y with $x \ne y$. Because $(x^2 - y^2)^2 + (2xy)^2 = n^2$, we see that n is the hypotenuse of a Pythagorean triple.

Also solved by, JPV Abad, Michel Bataille (France), Brian D. Beasley, Jean Bogaert (Belgium), Grady Bullington, Robert Calcaterra, John Christopher, Con Amore Problem Group (Denmark), Charles R. Diminnie, Fejéntaláltuka Szeged Problem Solving Group (Hungary), L. L. Foster, Elana C. Greenspan, Douglas Iannucci (Virgin Islands), Elias Lampakis (Greece), David P. Lang, Eugene Lee, S. C. Locke, David Lovit, David E. Manes, José H. Nieto (Venezuela), Northwestern University Math Problem Solving Group, Don Redmond, Nicholas C. Singer, Marian Tetiva (Romania), Paul Weisenhorn (Germany), and the proposer. Roy Barbara (Lebanon) solved part a. There were two incorrect submissions.

A complex inequality — February 2006

1738. *Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Baracelona, Spain.*

Let n be a positive number and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers. Prove that

$$\operatorname{Re}\left(\sum_{k=1}^{n} a_k b_k\right) \leq \frac{1}{2n}\left(\sum_{k=1}^{n} |a_k|^2 + \frac{4n^2-1}{3}\sum_{k=1}^{n} |b_k|^2\right).$$

Solution by Robert Calcaterra University of Wisconsin Platteville, Platteville, WI.

If a and b are real and n is a positive integer, then

$$(a-nb)^2 + \frac{(n^2-1)b^2}{3} \geq 0.$$

Expanding the expression on the left and solving for ab yields

$$ab \leq \frac{1}{2n}\left(a^2 + (4n^2-1)\frac{b^2}{3}\right). \qquad (*)$$

Because

$$\operatorname{Re}\left(\sum_{k=1}^{n} a_k b_k\right) \leq \left|\sum_{k=1}^{n} a_k b_k\right| \leq \sum_{k=1}^{n} |a_k||b_k|,$$

the desired inequality follows immediately from $(*)$.

Also solved by Michel Bataille (France), Jean Bogaert (Belgium), Con Amore Problem Group (Denmark), Prithwijit De (Ireland), Robert L. Doucette, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Dmitry Fleischman, Elias Lampakis (Greece), Tom Leong, John Mangual, José H. Nieto (Venezuela), Gabriel T. Prăjitură, Tony Tam, Paul Weisenhorn (Germany), and the proposer.

First quadrant paths — February 2006

1739. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.*

An object moves in the plane, starting from the origin, and at each step moving one unit up, down, to the right, or to the left. Find the number of such paths that stay in the quadrant $\{(x, y) : x, y \geq 0\}$, and consist of a total of n steps, exactly k of which are vertical (up or down).

Solution by David Lovit, student, Swarthmore College, Swarthmore, PA.

First let $k = 0$, so the path is on the x-axis and is never to the left of the origin. Let a_n be the number of such paths. We find a recursive formula for a_n. If $n-1$ is odd then a path of length $n-1$ cannot end at the origin, so a path of length n can be obtained by appending a left move or a right move to the end of the path. Thus, if n is even, then $a_n = 2a_{n-1}$. If $n-1$ is even, then the number of paths of length $n-1$ that end at the origin is the number of Dyke paths of order $\frac{n-1}{2}$, that is $\frac{2}{n+1}\binom{n-1}{(n-1)/2}$. Such paths can be made into paths of length n by appending a right move (but not a left move) to the end. Thus for $n \geq 2$,

$$a_n = \begin{cases} 2a_{n-1} & n \text{ even} \\ 2a_{n-1} - \dfrac{2}{n+1}\dbinom{n-1}{(n-1)/2} & n \text{ odd.} \end{cases}$$

Next we show that $a_n = \binom{n}{\lceil n/2 \rceil}$. This is clearly true for $n = 1$. If n is even, then

$$\binom{n}{\lceil n/2 \rceil} = \binom{n}{n/2} = 2\binom{n-1}{\lceil (n-1)/2 \rceil}.$$

If n is odd, then

$$\binom{n}{\lceil n/2 \rceil} = \binom{n}{(n+1)/2} = \frac{2n}{n+1}\binom{n-1}{(n-1)/2}$$
$$= 2\binom{n-1}{\lceil (n-1)/2 \rceil} - \frac{2}{n+1}\binom{n-1}{(n-1)/2}.$$

This proves that the proposed for formula for a_n satisfies the recursion and has the correct initial condition. Hence the formula is correct. For the given problem, we have a horizontal (left/right) path of length $n-k$ and a vertical (up/down) path both of which avoid negative values. Because we can choose the positions for the vertical moves in $\binom{n}{k}$ ways, the desired number of paths is

$$\binom{n}{k} a_k a_{n-k} = \binom{n}{k}\binom{k}{\lceil k/2 \rceil}\binom{n-k}{\lceil (n-k)/2 \rceil}.$$

Also solved by, JPV Abad, Michael Andreoli, Rich Avery, Michel Bataille (France), Robert Bernstein, Jany C. Binz (Switzerland), Jean Bogaert (Belgium), Grady Bullington, Robert Calcaterra, Con Amore Problem Group (Denmark), Knut Dale (Norway), Robert L. Doucette, Fejéntaláltuka Szeged Problem Solving Group (Hungary), G.R.A.20 Problem Solving Group (Italy), Leon Harkleroad, Frank Jurjevich, Tom Leong, S. C. Locke, Kim McInturff, William Moser (Canada), José H. Nieto (Venezuela), Northwestern University Math Problem Solving Group, Raúl Simón (Chile), R. S. Tiberio, Paul Weisenhorn (Germany), and the proposer. There was one incorrect submission.

$\mathbb{R}$ as a complex vector space — February 2006

1740. *Proposed by Michel Bataille, Rouen, France.*

Prove or disprove: there exists a scalar multiplication $(z, r) \to z * r$ from $\mathbb{C} \times \mathbb{R}$ into $\mathbb{R}$ such that $(\mathbb{R}, +, *)$ is a vector space over $\mathbb{C}$.

Solution by Miguel A. Lerma, Northwestern University, Evanston, IL.

We prove that in ZFC (Zermelo-Frankel set theory with the Axiom of Choice) the answer is yes. First note that $\mathbb{C}$ and $\mathbb{R}$ are both vector spaces over $\mathbb{Q}$ with (Hamel) bases of the same dimension, $2^{\aleph_0}$. Thus they are isomorphic as vector spaces over $\mathbb{Q}$, and consequently, $(\mathbb{C}, +)$ and $(\mathbb{R}, +)$ are group isomorphic. Because $\mathbb{C}$ is a one dimensional complex vector space, any group isomorphic to $(\mathbb{C}, +)$ will also be a one dimensional complex vector space. Given a group isomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}, +)$, a scalar multiplication $\mathbb{C} \times \mathbb{R} \to \mathbb{R}$ can be defined by

$$(z, x) \mapsto z * x = f^{-1}(zf(x)),$$

where the product $zf(x)$ is the usual complex multiplication.

In fact (in ZFC) even more is true. With a suitable scalar multiplication, $(\mathbb{R}, +)$ can be viewed as a complex vector space of any dimension $d \leq 2^{\aleph_0}$. This is because $\mathbb{C}^{(d)}$, the set of elements of $\mathbb{C}^d$ with only finitely many nonzero components, is also a vector space over $\mathbb{Q}$ of dimension $2^{\aleph_0}$. Thus $(\mathbb{C}^{(d)}, +)$ and $(\mathbb{R}, +)$ are group isomorphic. Because $\mathbb{C}^{(d)}$ is a d dimensional complex vector space over $\mathbb{C}$, the same is true of $\mathbb{R}$. Given a group isomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}^{(d)}, +)$ an inner product $\mathbb{C} \times \mathbb{R} \to \mathbb{R}$, can be defined by

$$(z, x) \mapsto z * x = f^{-1}(zf(x)),$$

where $zf(x)$ denotes the usual scalar product of an element $z \in \mathbb{C}$ with a complex vector $f(x) \in \mathbb{C}^{(d)}$.

This proof cannot be carried out in ZF without AC (the Axiom of Choice.) In fact, there are models of ZF without AC in which the additive groups of $(\mathbb{C}, +)$ and $(\mathbb{R}, +)$ are *not* isomorphic. For example, see the paper by C. J. Ash, "A Consequence of the Axiom of Choice", *J. Austral. Math. Soc.*, **19** (Series A) (1975), 306–308. In such models of ZF, $\mathbb{R}$ cannot be a one dimensional vector space over $\mathbb{C}$. More specifically Ash proves that in ZF, the assumption $(\mathbb{C}, +) \cong (\mathbb{R}, +)$ implies that there is a set of real numbers that is not Lebesgue measurable. Thus, in any model of ZF in which all subsets of $\mathbb{R}$ are Lebesgue measurable, $(\mathbb{C}, +) \cong (\mathbb{R}, +)$ cannot hold. Ash's reasoning also shows that in these models we cannot have $(\mathbb{C}^{(d)}, +) \cong (\mathbb{R}, +)$ for any $d \leq \aleph_0$.

Also solved by, Roy Barbara (Lebanon), Robert Calcaterra, Jim Delany, and the proposer. There were six incomplete or incorrect submissions.

Answers

Solutions to the Quickies from page 78.

A967. Let F be a real valued function define on I with $F'(x) = f(x)$ for all $x \in I$. Let $\epsilon > 0$ be given. Then for each $x \in I$ there is a $\delta(x) > 0$ such that if $y \in I$ and $0 < |x - y| < \delta(x)$, then

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \frac{\epsilon}{2}.$$

Now let $x, y \in I$ with $|x - y| < \delta(x)$ and $|x - y| < \delta(y)$. If $x = y$, then $|f(x) - f(y)| < \epsilon$ is immediate. If $x \neq y$, then

$$\begin{aligned} |f(x) - f(y)| &= \left| f(x) - \frac{F(y) - F(x)}{y - x} + \frac{F(x) - F(y)}{x - y} - f(y) \right| \\ &\leq \left| f(x) - \frac{F(y) - F(x)}{y - x} \right| + \left| f(y) - \frac{F(x) - F(y)}{x - y} \right| < \epsilon. \end{aligned}$$

A968. First observe that

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = \frac{\ln a}{\ln abc} + \frac{\ln b}{\ln abc} + \frac{\ln c}{\ln abc} = 1.$$

Because $\log_a abc$, $\log_b abc$, and $\log_c abc$ are all positive, we can apply the arithmetic-harmonic mean inequality to obtain

$$\frac{\log_a abc + \log_b abc + \log_c abc}{3} \geq \frac{3}{\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}} = 3,$$

and the inequality follows. Equality occurs if and only if $\log_a abc = \log_b abc = \log_c abc$, which is the case if and only if $a = b = c$.

A similar result holds if $0 < a, b, c < 1$, and there is a natural extension to more than three variables.

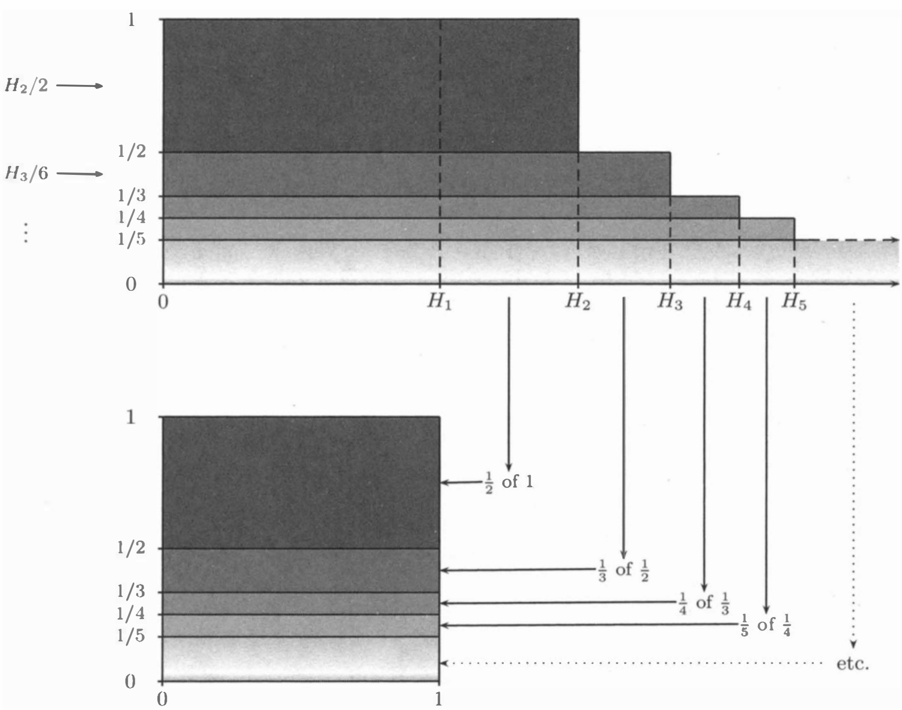
Proof Without Words: A Series Involving Harmonic Sums

Given

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

show that

$$\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} = \frac{H_2}{2} + \frac{H_3}{6} + \frac{H_4}{12} + \frac{H_5}{20} + \cdots = 2$$



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REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Hawking, Stephen (ed.), *God Created the Integers: The Mathematical Breakthroughs that Changed History*, Running Press, 2005; xiii + 1162 pp, $29.95. ISBN 0-7624-1922-9.

This is not a coffee-table book: It doesn't have enough pictures, they aren't in color, and the book's shape is all wrong. It is instead a welcome compendium of (translated) selections from the works of a few famous mathematicians throughout the ages: Euclid, Archimedes, Diophantus, Descartes, Newton, Laplace, Fourier, Gauss, Cauchy, Boole, Riemann, Weierstrass, Dedekind, Cantor, Lebesgue, Gödel, and Turing. (Curiously, Kronecker, from whose famous quotation came the title of the book, didn't make the cut.) The book differs from previous source books on mathematics (such as those edited by Smith, Struik, and Calinger) in providing vastly more text each from considerably fewer individuals. Editor Hawking provides accompanying biographical sketches; but the connections hinted at in the subtitle ("changed history") are not argued or elaborated, even with "history" restricted to the less-encompassing (and less-marketable) "history of mathematics." Nevertheless, it is noteworthy and regrettable that most mathematics majors graduate having read only textbooks and no original works at all; a collection like this one could form the basis for a fruitful senior seminar.

Szpiro, George G., *The Secret Life of Numbers: 50 Easy Pieces on How Mathematicians Work and Think*, Joseph Henry Press, 2006; xii + 210 pp, $24.95. ISBN 0-309-09658-8.

The world has only a handful or two of mathematical journalists; George Szpiro, a mathematical economist and author of *Kepler's Conjecture: How Some of the Greatest Minds in History Helped Solve One of the Oldest Math Problems in the World* (2003), writes a column for the Swiss newspaper *Neue Zürcher Zeitung* and won a €5,000 prize for it. His columns from the past several years, collected here in translation, read with the requisite lightness and interest, whether treating the mathematically mundane (leap years), unsolved conjectures (the Poincaré conjecture and others), personalities (Abel, Bernays, Coxeter, Wolfram), or applications (market efficiency, Internet servers, Bible codes). Szpiro includes references, and the book has an index.

Derbyshire, John, *Unknown Quantity: A Real and Imaginary History of Algebra*, Joseph Henry Press, 2006; viii + 374 pp, $27.95. ISBN 0-309-09657-X.

In 2005 there were only 17,000 bachelor's degrees awarded in mathematics by U.S. colleges, about 1% of all bachelor's degrees. That is remarkably few, given the exciting discoveries in mathematics in the past decade and the enormous flood of expository and popular literature about mathematics. This book demands only basic algebra from its reader; but it offers a journey that spans both the centuries leading up to literal notation and the discoveries and abstraction that take the reader through developments of the twentieth century. It is a book that should be in every high school library and public library.

Klemens, Ben, *Math You Can't Use: Patents, Copyrights, and Software*, Brookings Institution Press, 2006; ix + 181 pp, $28.95. ISBN-13: 978-0-8157-4942-4; ISBN-10: 0-8157-4942-2.

Well, you can use it, but you have to pay, even if it's fairly obvious and you thought it up yourself. Despite a longstanding holding that a mathematical equation is not patentable (being a law of nature, which no person can own), the U.S. Patent Office has been granting patents for algorithms, making the situation worse through poor decisions about what is "nonobvious," a requirement for a patent to be issued. The author traces the history of patentability of software (which he views as indistinguishable from mathematics, per the Church-Turing thesis) and makes recommendations for reform by Congress. (Did you know that Paramount Pictures claims to own the Klingon language of *Star Trek* and that you can't write the Great Klingon Novel in it without their permission?)

Nahin, Paul J., *When Least Is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible*, Princeton University Press, 2004; xviii + 370 pp, $34.95. ISBN 0-691-07078-4.

This book is a fascinating, engrossing, and inspiring introduction to optimization, for a reader who knows calculus. It follows history, starting from ancient extremal problems (Dido's problem) through medieval ones (maximizing visual angle for a painting) and on to calculus (Snell's law, wine barrels, projectile motion) and beyond (Steiner problem, traveling sales problem, linear programming, dynamic programming). The explanation of the paradox of Torricelli's funnel (finite volume, infinite area) is clever. Did you know that the Jensen of Jensen's inequality was a telephone engineer? "This book has been written from the practical point of view of the engineer," so it uses different plurals for "maximum" and "minimum" than most mathematicians do; but that's OK. The discussion of gunnery and basketball shooting ignores air resistance, so the figure on p. 159 should show a parabola. Unfortunately, the possessive of "Huygens" is not "Huygen's" (Fig. 1.1, p. 8); in the index, "Regiomontanus" appears as "Rigiomontanus," while "Erdős" appears there (and in the text) as "Erdoõs," and Torricelli's funnel is not to be found. Unfortunately, there is no collected bibliography for further reading, though citations occur throughout.

Wildberger, N. J., *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Pty Ltd (Australia), 2005; xx + 300 pp, $89.95. ISBN 0-9757492-0-X.

My elder son is currently spending a year in high school studying trigonometry, a length of time I consider somewhere between extravagant and excessive (though it will certainly take all year to get most of the way through the 500-page textbook). Author Wildberger offers an alternative approach to trigonometry whose basis is not Euclidean distance and traditional angle measure but "quadratic" concepts that are ratios of quadratic expressions in the coordinates. Converted to familiar terms, his *quadrance* and *spread* are the squares of Euclidean distance and sine of an angle. Square roots and transcendentals almost disappear. Moreover, this trigonometry transfers completely to any other number field (except characteristic 2) and hence offers "universality." The author claims that this new theory will take "less than half the usual time to learn"; but I doubt it, and it would still have to be interfaced with the traditional concepts and notation. A score of errata are at `http://wildegg.com/papers/FinalErrata.pdf`.

Gardner, Martin, *The Colossal Book of Short Puzzles and Problems*, Norton, 2006; xiv + 496 pp, $35. ISBN 0-393-06114-0. *Aha! A Two Volume Collection: Aha! Gotcha, Aha! Insight*, MAA, 2006; xvii + 164 pp, ix + 181 pp, $47.50 (member: $37.95). ISBN-10: 0-88385-551-8, ISBN-13: 978-0-88385-551-5.

Sudoku has whetted the public's appetite for mathematical puzzles once again. How long till Sudoku, like Rubik's Cube, runs its course? These collections of puzzles by Martin Gardner may be appearing at just the right time to direct the public to other forms of mathematical recreation and to Gardner's other marvelous exposition in mathematics.

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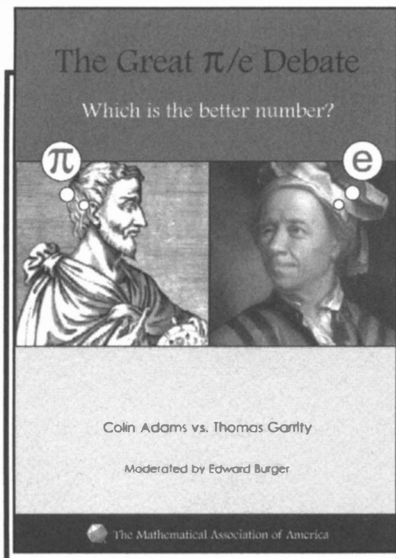

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